Daily Sequences Drill #6

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: BMO1 Capstone I — Constrained Integer Sequences · Extremal Construction · Alternating Recurrences.

Attempt each question before checking answers. Time yourself. No calculators.

Section A Rapid Recognition (≤15 s each)

Integer-mean conditions and fixed points on sight.

1. If the mean of the first 3 terms of a sequence is an integer, what can be said about $a_1 + a_2 + a_3$?

2. A sequence starts $a_1 = 5$. For the mean of the first two terms to be an integer, what parity must a_2 have?

3. Two players alternately write numbers; the second player always writes twice the first player's last number. What is the fixed point of the combined "doubling" operation?

4. A sequence has $a_{n+1} = 2a_n - 10$. Find its fixed point.

5. Recall Day 4's shift trick: express the solution to $a_{n+1} = 2a_n - 10$ as a GP shifted by its fixed point.

6. Give an example of 3 distinct positive integers whose average is an integer.

7. True or false: if n integers have an integer average, their sum must be a multiple of n.

8. A sequence of positive integers has running sum 20 after 4 terms. Is the mean of these 4 terms an integer?

9. Recall Day 5's fixed-point theorem: if $a_{n+1} = f(a_n)$ converges to L, then $f(L) = L$. If a sequence instead must eventually *repeat* a value (not necessarily converge), does $f(L) = L$ still have to apply at the repeated value?

10. A sequence of 5 integers has partial sums with $S_1 = 3, S_2 = 8$. Is S_2 divisible by 2? Is the mean of the first two terms an integer?

Section B Manipulation Drills (≤50 s each)

Construct sequences under a constraint; solve the alternating-recurrence fixed point.

1. Construct a sequence of 4 distinct positive integers such that every partial sum S_1, S_2, S_3, S_4 is divisible by its index.

2. Using your sequence from B1, if it is $1, 3, 5, 7$, what is $S_4/4$ (the mean of all four terms)?

A) 4

B) 16

C) 7

D) 2

3. Alice writes a, Bob writes $3a$, Alice writes $3a - 12$ (Bob's value minus 12), and this two-step pattern repeats (Bob triples, Alice subtracts 12). Find the fixed point of Alice's combined per-turn map $f(x) = 3x - 12$.

4. If $a = 6$ exactly (the fixed point from B3), what happens to every subsequent number written?

A) Every Alice-turn writes exactly 6 forever; Bob alternates writing 18.

B) The numbers grow without bound.

C) The numbers eventually reach 0.

D) The sequence is not well-defined.

5. For a general starting value a, express Alice's value after k applications of $f(x) = 3x - 12$ in closed form, using the shift-to-fixed-point trick.

A) $(a - 6)3^k + 6$

B) $a \cdot 3^k - 6$

C) $3^k a - 6k$

D) $(a + 6)3^k - 6$

6. Using B5's formula, show that Alice's OWN sequence of values (ignoring Bob's separately-written numbers) can only return to the original a again if a already equals the fixed point 6.

7. So if $a \neq 6$ is ever going to reappear, it must happen at one of Bob's turns. Bob's k-th value is $3\left[(a - 6)3^{k-1} + 6\right]$. Setting this equal to a and writing $u = a - 6$, which equation results?

A) $u(3^k - 1) = -12$

B) $u(3^k - 1) = 12$

C) $u(3^k + 1) = -12$

D) $u = 12(3^k - 1)$

8. Using B7's equation $u = \dfrac{-12}{3^k - 1}$, find all integer values of $k \geq 1$ giving integer u, and hence all possible values of $a = u + 6$ (besides the trivial $a = 6$).

9. In general, for the process "multiply by r, then subtract c," how many possible values of a can cause a to reappear at exactly Bob's first turn ($k = 1$)?

A) Exactly 1 candidate (from solving one linear equation), though it may not be an integer for given r, c.

B) Always 2.

C) Infinitely many.

D) It depends on the sign of a.

10. A sequence of 6 positive integers has every partial sum $S_1, \ldots, S_6$ divisible by its index. Given $a_1 = 2$, construct one valid choice for $a_2, \ldots, a_6$.

Section C Substitution & Structure (≤90 s each)

General principles behind both constructions.

1. Which of the following is a general construction of positive integers $a_1, a_2, \ldots, a_n$ (for any n) that ALWAYS makes every partial sum S_k divisible by k?

A) $a_k = k$ for all k.

B) $a_k = 2^k$ for all k.

C) $a_k = 1$ for all k.

D) $a_k = k^2$ for all k.

2. In the alternating "multiply by r, subtract c" process, the fixed point of Alice's per-turn map $f(x) = rx - c$ is always which formula (for $r \neq 1$)?

A) $x = \dfrac{c}{r-1}$

B) $x = \dfrac{c}{r+1}$

C) $x = c(r-1)$

D) $x = \dfrac{r}{c-1}$

3. For $r = 5, c = 20$, what is the fixed point?

A) 5

B) 4

C) 20

D) 25

4. Which is the correct general formula for Alice's fixed point (same setup as C2)?

A) $x = \dfrac{c}{r-1}$

B) $x = \dfrac{c}{r+1}$

C) $x = c(r-1)$

D) $x = \dfrac{r}{c-1}$

5. Values of a that cause a to reappear come from two sources: Alice's own fixed point, and (at most) finitely many solutions from Bob's-turn-matches-a equations. Which statement about the second source is always true?

A) It has infinitely many integer solutions in general.

B) It has at most one candidate value of a for each choice of k, but only finitely many k can give an integer solution, since $|u| = \frac{c}{r^k-1} \to 0$ as $k \to \infty$.

C) It never has any integer solutions.

D) It always has exactly one integer solution for every r, c.

6. A sequence of positive integers has S_n divisible by n for every n up to some large value. Given a_1 is known, which must be true about a_2?

A) $a_1 + a_2$ is even.

B) $a_2 = a_1$.

C) a_2 must be even.

D) a_2 must be a multiple of a_1.

7. A sequence of positive integers is built one at a time, maintaining S_n divisible by n at each step. At each step, does a valid next term always exist?

A) Yes — choosing $a_{n+1} \equiv -S_n \pmod{n+1}$ (the least positive such residue, or any larger value with that residue) always gives a valid S_{n+1}.

B) No — eventually the sequence must get stuck.

C) It depends on whether n is prime.

D) Only if all previous terms were even.

8. Given an UNLIMITED supply of positive integers to choose from (no finite deck), must the construction from C7 eventually get stuck?

A) No — with unlimited supply the process can always continue forever; getting stuck is only possible with a FINITE pool of available numbers.

B) Yes, it must get stuck within finitely many steps regardless of supply.

C) It depends on the value of a_1.

D) The process is stuck immediately after the first step.

Section D Challenge (SMC / BMO1 difficulty)

Full BMO1-style proofs. Insight over computation — write the argument out in full.

1. A deck contains cards numbered 1 to 6. Cards are drawn one at a time and placed in a row (each new card added to the right), maintaining that at every stage the mean of the numbers placed so far is an integer. Play stops when no remaining card can be validly added. Prove that the smallest possible number of cards in the row when play stops is exactly 3, and give an example achieving it. *(after BMO1 2018 Q6)*

2. Alice and Bob take turns writing numbers on a board. Alice starts by writing an integer a with $-100 \le a \le 100$. On each of Bob's turns, he writes twice the number Alice wrote last. On each of Alice's subsequent turns, she writes the number 45 less than the number Bob wrote last. At some point, the number a is written on the board for a second time. Find all possible values of a. *(after BMO1 2021 Q1)*

3. Prove, in general, that for the alternating process with integer ratio $r \ge 2$ and positive integer subtraction constant c, a starting value $a \ne \frac{c}{r-1}$ can cause a to reappear at Bob's k-th turn for at most finitely many values of k — and hence that the full search for all valid a always terminates.

4. For the card-sequencing problem (as in D1), with cards numbered 1 to N: is it always true that at least 2 cards can be placed (i.e. length-1-stuck is impossible) whenever $N \ge 4$?

A) True — for $N \ge 4$, both parity classes have at least 2 members, so removing any single card always leaves at least one same-parity card remaining.

B) False — length-1-stuck remains possible for some $N \ge 4$.

C) It depends on whether N is prime.

D) True only for even N.

5. In both the card-sequencing problem (D1) and the alternating-writing problem (D2), the eventual stopping/repeating behaviour was forced by a finiteness argument — a finite pool of cards, or a shrinking magnitude $|u| \to 0$. What is the common underlying principle?

A) A process constrained to a finite or shrinking set of possibilities cannot continue satisfying an ever-tightening condition forever — it must eventually terminate or repeat.

B) Every recurrence relation eventually becomes periodic.

C) Integer sequences are always bounded.

D) There is no common principle; the two problems are unrelated.

"If your BMO1 solution ran to four pages, you have found an answer, not the answer."

Found a mistake or want to contribute a sheet?
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