# Daily Sequences Drill #5 — Answers & Investigations

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| Section | Topic | Questions | Target time |
|---|---|---|---|
| A | Rapid Recognition | 10 | 2 min 30 sec |
| B | Manipulation Drills | 10 | 8 min |
| C | Substitution & Structure | 8 | 10 min |
| D | Challenge | 5 | 15 min |

*New toolkit today: Monotonicity & Boundedness · Fixed Points · Non-Modular Periodicity · Floor/Ceiling Sequences.*

## Section A Rapid Recognition

**1.** Is the sequence $a_n = 5 - n$ increasing, decreasing, or neither?

**Answer:** Decreasing

**Method:** $a_{n+1} - a_n = (5 - (n+1)) - (5 - n) = -1 < 0$ for every $n$.

> **Investigate further:** Every AP with $d < 0$ decreases — Day 1's definition in new language. **Is this sequence bounded? Decide above and below separately, and say which of the two fails.** Then state the general rule: for an AP, exactly which of "bounded above" and "bounded below" can hold, and what does $d = 0$ do to the answer?

**2.** Is the sequence $a_n = n^2$ bounded above? Bounded below?

**Answer:** Not bounded above; bounded below (by 1).

**Method:** $n^2 \to \infty$ as $n \to \infty$, so no upper bound exists. But $n^2 \geq 1$ for every positive integer $n$, so 1 is a lower bound.

> **Investigate further:** A sequence can be bounded on one side only, so check each direction separately. **Now find a sequence that is bounded above but not below, and one bounded below but not above — then decide whether a sequence bounded on *neither* side must be non-monotonic.** The last one is the interesting case; try to build a counterexample before assuming.

**3.** Find the fixed point of $f(x) = 2x - 3$ (i.e. solve $f(x) = x$).

**Answer:** $x = 3$

**Method:** $x = 2x - 3 \Rightarrow -x = -3 \Rightarrow x = 3$.

> **Investigate further:** Check $f(3) = 3$ ✓: once the sequence lands on 3 it stays forever. **But is** 3 **attracting or repelling? Start at** $a_1 = 3.1$ **and iterate a few times, then try** $a_1 = 2.9$. Work out what $|f'(x)| = 2$ predicts, and hence why almost every starting value runs *away* from this fixed point.

**4.** A sequence satisfies $a_{n+1} = \dfrac{1}{a_n}$. If $a_1 = 2$, find $a_2, a_3, a_4$.

**Answer:** $a_2 = \frac{1}{2}, a_3 = 2, a_4 = \frac{1}{2}$

**Method:** $a_2 = \frac{1}{a_1} = \frac{1}{2}$. $a_3 = \frac{1}{a_2} = 2$. $a_4 = \frac{1}{a_3} = \frac{1}{2}$.

> **Investigate further:** Period 2: odd terms 2, even terms $\frac{1}{2}$, forever. **Which starting values $a_1$ make this sequence constant rather than genuinely period-2?** Solve $f(x) = x$ for $f(x) = \frac{1}{x}$ — and note there are two answers, which is already unlike the linear map in A3.

**5.** Compute $\lfloor 3.7 \rfloor$ and $\lceil 3.2 \rceil$.

**Answer:** $\lfloor 3.7 \rfloor = 3$, $\lceil 3.2 \rceil = 4$

**Method:** Floor rounds down to the nearest integer; ceiling rounds up.

> **Investigate further:** Both operations fix an integer: $\lfloor 5 \rfloor = \lceil 5 \rceil = 5$. **So for which real $x$ is $\lceil x \rceil - \lfloor x \rfloor = 0$, and what is it for every other $x$?** The function takes only two values ever — state both cases precisely, and be careful about negative $x$.

**6.** Recall from Day 4 that a recurrence with both characteristic roots inside the unit circle converges to 0. Does $a_n = \left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n$ converge as $n \to \infty$? To what value?

**Answer:** Yes, converges to 0.

**Method:** Both $\left(\frac{1}{2}\right)^n \to 0$ and $\left(\frac{1}{3}\right)^n \to 0$ as $n \to \infty$, since both bases have magnitude less than 1. Their sum also $\to 0$.

> **Investigate further:** Day 4's D5 with $A = B = 1$ and both roots inside the unit circle. **Which of the two terms dies away faster, and roughly how many steps until the $\left(\frac{1}{3}\right)^n$ part is negligible against the $\left(\frac{1}{2}\right)^n$ part?** Form their ratio and watch it shrink — this is why the *largest* root governs long-run behaviour.

**7.** Is the sequence $a_n = (-1)^n$ bounded? Does it converge?

**Answer:** Bounded (yes); does not converge.

**Method:** Every term is either 1 or $-1$, so $-1 \le a_n \le 1$ always (bounded). But the sequence never settles near a single value — it alternates forever.

> **Investigate further:** The standard counterexample to "bounded implies convergent". **Sharpen it: this sequence has two subsequences that each converge, to different limits. Identify them, and explain why that is exactly what stops the whole sequence converging.** A limit, if it exists, must be shared by every subsequence.

**8.** True or false: every bounded sequence converges.

**Answer:** False

**Method:** $a_n = (-1)^n$ (A7) is a direct counterexample: bounded, but not convergent.

> **Investigate further:** The correct statement pairs bounded *with* monotonic; boundedness alone is not enough. **Check the converse before trusting it — must every convergent sequence be monotonic?** Find a convergent sequence that is not, and then say which of the two implications is actually true.

**9.** Is the sequence $a_n = \dfrac{n}{n+1}$ increasing or decreasing? (Check $a_1, a_2, a_3$.)

**Answer:** Increasing

**Method:** $a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, a_3 = \frac{3}{4}$ — each term is bigger than the last.

> **Investigate further:** B1 proves this for all $n$, not just the first three terms. **Before turning the page: three terms establish a pattern but never a proof — construct a sequence that increases for its first three terms and then decreases.** That is exactly why B1 has to do algebra rather than arithmetic.

**10.** Find the smallest $n$ for which $\lfloor \sqrt{n} \rfloor = 4$.

**Answer:** 16

**Method:** $\lfloor \sqrt{n} \rfloor = 4$ needs $4 \le \sqrt{n} < 5$, i.e. $16 \le n < 25$. The smallest such $n$ is 16.

> **Investigate further:** Convert a floor equation into a double inequality on $n$ before searching. **Do it in general: for which $n$ is $\lfloor \sqrt{n} \rfloor = k$, and how many such $n$ are there for each $k$?** Your count should come out as a simple linear expression in $k$ — which is precisely what C5 then adds up.

## Section B Manipulation Drills

**1.** Prove that $a_n = \dfrac{n}{n+1}$ is increasing for all $n \ge 1$ by showing $a_{n+1} - a_n > 0$.

**Answer:** Proof: $a_{n+1} - a_n = \dfrac{1}{(n+1)(n+2)} > 0$.

**Method:** $a_{n+1} - a_n = \dfrac{n+1}{n+2} - \dfrac{n}{n+1} = \dfrac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \dfrac{n^2 + 2n + 1 - n^2 - 2n}{(n+2)(n+1)} = \dfrac{1}{(n+1)(n+2)}$, which is positive for every $n \ge 1$. $\square$

> **Investigate further:** The "common denominator, then simplify" route to $a_{n+1} - a_n > 0$ works on almost any rational sequence. **Practise it on $a_n = \frac{2n+1}{n+3}$: is it increasing or decreasing, and what is its limit?** Compute the difference as a single fraction and check only the numerator's sign — the denominator is positive throughout.

**2.** Is $a_n = \dfrac{n}{n+1}$ bounded above? If so, by what value (the supremum)?

A) Yes, supremum 1.

B) Yes, supremum 2.

C) No, unbounded.

D) Yes, supremum $\frac{1}{2}$.

**Answer:** A) Yes, supremum 1.

**Method:** $a_n = \dfrac{n}{n+1} = 1 - \dfrac{1}{n+1} < 1$ for every $n$, and $a_n \to 1$ as $n \to \infty$ — so 1 is the smallest possible upper bound (the supremum), even though no term ever equals it exactly.

> **Investigate further:** Rewriting $\frac{n}{n+1}$ as $1 - \frac{1}{n+1}$ makes both the increase and the bound obvious at a glance. **Is that supremum 1 ever actually attained? And is there a smaller upper bound that works?** Distinguish "an upper bound" from "the least upper bound" — the sequence never reaches 1, yet nothing below 1 bounds it.

**3.** A sequence satisfies $a_{n+1} = \dfrac{1}{a_n}$ with $a_1 = 3$. Find $a_{100}$.

A) 3

B) $\frac{1}{3}$

C) 100

D) $\frac{1}{100}$

**Answer:** B) $\frac{1}{3}$

**Method:** The sequence has period 2: $a_1 = 3, a_2 = \frac{1}{3}, a_3 = 3, a_4 = \frac{1}{3}, \ldots$ — odd indices give 3, even indices give $\frac{1}{3}$. Since 100 is even, $a_{100} = \frac{1}{3}$.

> **Investigate further:** Reduce a large index modulo the period first: $100 \equiv 0$, the even case. **Now handle the general reciprocal map: for $a_{n+1} = \frac{1}{a_n}$ with any $a_1 \neq 0$, write $a_n$ in closed form using the parity of $n$.** Then say which starting values collapse the period-2 pattern to a constant sequence.

**4.** Find the fixed point(s) of $f(x) = \dfrac{1}{x}$ (i.e. solve $f(x) = x$).

**Answer:** $x = 1, -1$

**Method:** $\dfrac{1}{x} = x \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

> **Investigate further:** $f(x) = \frac{1}{x}$ has two fixed points, unlike the linear map in A3 which has exactly one. **Explain why any non-identity linear $f(x) = mx + c$ can have at most one — and identify the single case where it has infinitely many.** Solve $mx + c = x$ and look hard at what makes that equation degenerate.

**5.** A sequence satisfies $a_{n+1} = f(a_n)$ where $f$ has a fixed point at $x = 5$. If $a_1 = 5$, what is $a_n$ for every $n$?

A) $a_n = 5$ for all $n$.

B) $a_n = 5n$.

C) $a_n \to 5$ but never equals it.

D) Cannot be determined.

**Answer:** A) $a_n = 5$ for all $n$.

**Method:** $a_1 = 5$ is exactly the fixed point, so $a_2 = f(5) = 5$, and by the same reasoning every later term stays at 5 forever.

> **Investigate further:** The simplest case of C8: start at a fixed point and you never leave. **But is the converse true — if a sequence is eventually constant, must that constant be a fixed point of $f$?** Consider what $a_{n+1} = f(a_n)$ forces once two consecutive terms agree.

**6.** Which of the following sequences is monotonic (always increasing or always decreasing)?

A) $a_n = (-1)^n$

B) $a_n = \sin(n)$

C) $a_n = \dfrac{1}{n}$

D) $a_n = n \bmod 3$

**Answer:** C) $\dfrac{1}{n}$

**Method:** $\dfrac{1}{n}$ strictly decreases as $n$ increases. A) and B) oscillate without settling into one direction; D) cycles through $1, 2, 0, 1, 2, 0, \ldots$ forever.

> **Investigate further:** Monotonic means *every* consecutive pair moves the same way; one exception disqualifies it. **Prove the claim that lurks behind this: show a periodic sequence can only be monotonic if it is constant.** Suppose it increases somewhere, then use periodicity to return to an earlier value and derive a contradiction.

**7.** Evaluate $\lfloor -2.3 \rfloor$.

A) $-2$

B) $-3$

C) $2$

D) $3$

**Answer:** B) $-3$

**Method:** Floor always rounds toward $-\infty$, not toward 0: the greatest integer that is still $\leq -2.3$ is $-3$, not $-2$.

> **Investigate further:** The classic slip: for negatives, "round down" means *more* negative, not toward zero. **So is $\lfloor -x \rfloor = -\lfloor x \rfloor$ always, sometimes, or never true?** Test it on an integer and a non-integer, then write down the correct identity relating $\lfloor -x \rfloor$ to $\lceil x \rceil$.

**8.** A sequence is defined by $a_n = \left\lfloor \dfrac{n}{2} \right\rfloor$. Find $a_1, a_2, a_3, a_4, a_5$.

A) $0, 1, 1, 2, 2$

B) $1, 1, 2, 2, 3$

C) $0, 0, 1, 1, 2$

D) $0, 1, 2, 2, 3$

**Answer:** A) $0, 1, 1, 2, 2$

**Method:** $a_1 = \lfloor 0.5 \rfloor = 0$, $a_2 = \lfloor 1 \rfloor = 1$, $a_3 = \lfloor 1.5 \rfloor = 1$, $a_4 = \lfloor 2 \rfloor = 2$, $a_5 = \lfloor 2.5 \rfloor = 2$.

> **Investigate further:** Floor division compresses the sequence, repeating each value twice. **Generalise: for $a_n = \lfloor \frac{n}{k} \rfloor$, how many times does each value repeat, and what is the smallest $n$ with $a_n = m$?** Check your formula against $k = 2$ here before trusting it.

**9.** Determine whether $a_n = \dfrac{(-1)^n}{n}$ converges, and if so, to what limit.

**Answer:** Converges to 0.

**Method:** $\left| \dfrac{(-1)^n}{n} \right| = \dfrac{1}{n} \to 0$ as $n \to \infty$, and since $-\dfrac{1}{n} \leq a_n \leq \dfrac{1}{n}$, the sequence is squeezed to 0 regardless of the alternating sign.

> **Investigate further:** Unlike $(-1)^n$, here the magnitude shrinks as the sign alternates, so the oscillation dies out. **Is this sequence monotonic? And does it therefore contradict the "bounded and monotonic" theorem from A8?** Answer both carefully — the theorem gives a sufficient condition, not a necessary one, and that distinction is the whole point.

**10.** For an AP with common difference $d$, under what condition on $d$ is the sequence strictly increasing?

A) $d > 0$

B) $d < 0$

C) $d \neq 0$

D) $d \geq 0$

**Answer:** A) $d > 0$

**Method:** $a_{n+1} - a_n = d$ for every $n$ in an AP — strictly increasing means this difference must be positive, i.e. $d > 0$.

> **Investigate further:** Day 1's fact in this sheet's language. **Now combine the two ideas: can an AP ever be both strictly increasing *and* bounded above?** Argue from the $n$th-term formula rather than intuition — and note this is exactly why the "bounded and monotonic" theorem never applies to a non-constant AP.

## Section C Substitution & Structure

**1.** A sequence satisfies $a_{n+1} = \dfrac{a_n - 1}{a_n + 1}$, with $a_1 = 2$. Find $a_5$.

A) 2

B) $\frac{1}{3}$

C) $-\frac{1}{2}$

D) $-3$

**Answer:** A) 2

**Method:** $a_1 = 2$. $a_2 = \dfrac{2-1}{2+1} = \dfrac{1}{3}$. $a_3 = \dfrac{1/3 - 1}{1/3 + 1} = \dfrac{-2/3}{4/3} = -\dfrac{1}{2}$. $a_4 = \dfrac{-1/2 - 1}{-1/2 + 1} = \dfrac{-3/2}{1/2} = -3$. $a_5 = \dfrac{-3-1}{-3+1} = \dfrac{-4}{-2} = 2$.

> **Investigate further:** Exact period 4: the cycle $2, \frac{1}{3}, -\frac{1}{2}, -3$ repeats forever. **Is that period a property of the starting value or of the map itself? Try $a_1 = 5$ and see whether you again return after exactly 4 steps.** Then find the one starting value in this cycle's orbit that the map cannot accept at all.

**2.** Continuing C1's sequence, what is $a_{101}$?

A) 2

B) $\frac{1}{3}$

C) $-\frac{1}{2}$

D) $-3$

**Answer:** A) 2

**Method:** The sequence has period 4 (from C1). $101 = 4(25) + 1$, so $101 \equiv 1 \pmod 4$, meaning $a_{101} = a_1 = 2$.

> **Investigate further:** Same index reduction as B3, now with period 4. **State the rule in general: if a sequence has period $p$, which term equals $a_N$, and what is the right convention when $N \equiv 0 \pmod p$?** Off-by-one errors here are the main hazard — test your rule on $N = 4, 5$ against the cycle you already know.

**3.** A sequence satisfies $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 2}$. Which is true about $(a_n)$?

A) Increasing and bounded above by 2; converges to 2.

B) Decreasing and bounded below by 1.

C) Not monotonic; oscillates.

D) Unbounded, diverges to infinity.

**Answer:** A) Increasing and bounded above by 2; converges to 2.

**Method:** Computing terms: $a_1 = 1, a_2 = \sqrt{3} \approx 1.73, a_3 \approx 1.93, a_4 \approx 1.98, \ldots$ — visibly increasing and approaching 2. (D1 proves this rigorously.)

> **Investigate further:** Solving $x = \sqrt{x+2}$ gives $x^2 - x - 2 = 0$, so $x = 2$ (rejecting $x = -1$, since $f \geq 0$). **Squaring can introduce false roots — so check directly why $-1$ fails, then investigate whether the limit depends on the start: try $a_1 = 100$ and $a_1 = 0$.** Does every non-negative starting value reach the same limit?

**4.** The first seven terms of a sequence of positive integers are: $w_1 = 10, w_2 = 15, w_3 = 18, w_4 = 22, w_5 = 31, w_6 = 36, w_7 = 41$. Consider the statement $(*)$: "If $n$ is prime, then $w_n$ is a multiple of 3 or a multiple of 5." What is the smallest value of $n$ that provides a counterexample to $(*)$? *(after TMUA 2021 Paper 2 Q10)*

A) 2

B) 3

C) 5

D) 7

**Answer:** C) 5

**Method:** Check the primes in order: $n = 2$: $w_2 = 15 = 3 \times 5$, a multiple of both — not a counterexample. $n = 3$: $w_3 = 18 = 3 \times 6$, a multiple of 3 — not a counterexample. $n = 5$: $w_5 = 31$, which is neither a multiple of 3 ($31 = 3 \times 10 + 1$) nor of 5 — this IS a counterexample. Since 5 is the smallest prime that fails, it's the answer.

> **Investigate further:** Check the candidate indices in increasing order and stop at the first failure rather than testing every term. **Having found the smallest counterexample, ask whether it is the only one — are there further indices where the claim also fails? Then try the real TMUA version (2021 Paper 2 Q10)**: sequence $15, 21, 30, 37, 44, 51, 59$, claim "multiple of 3 or 5", which also breaks first at $n = 5$.

**5.** A sequence is defined by $a_n = \lfloor \sqrt{n} \rfloor$. For how many values of $n$ in the range $1 \le n \le 100$ does $a_n = 7$?

A) 13

B) 14

C) 15

D) 16

**Answer:** C) 15

**Method:** $a_n = 7$ needs $7 \le \sqrt{n} < 8$, i.e. $49 \le n < 64$. The integers from 49 to 63 inclusive number $63 - 49 + 1 = 15$.

> **Investigate further:** A10's boundary technique extended to counting a range. **Push it to the full sum: over $1 \le n \le 100$, how many $n$ give each value of $\lfloor \sqrt{n} \rfloor$, and do your counts add to exactly 100?** That total is the check that catches an off-by-one at the top end, where the last block is cut short.

**6.** A sequence is defined by $a_n = n - \lfloor n/3 \rfloor \cdot 3$ (the remainder when $n$ is divided by 3, written using floor notation). Is $(a_n)$ periodic? If so, what is its period?

A) Periodic, period 3.

B) Periodic, period 2.

C) Not periodic.

D) Periodic, period 1 (constant).

**Answer:** A) Periodic, period 3.

**Method:** $a_n = n - 3\lfloor n/3 \rfloor$ is exactly the remainder of $n$ divided by 3: for $n = 1, 2, 3, 4, 5, 6, \ldots$ this gives $1, 2, 0, 1, 2, 0, \ldots$ — repeating every 3 terms.

> **Investigate further:** $n \bmod 3$ and $n - 3\lfloor n/3 \rfloor$ are the same formula written twice. **This sequence is periodic — state its period and its repeating block, then decide whether the identity still holds for negative $n$.** Python and most mathematicians agree on negative $n$ here; some calculators do not, which is worth knowing before you trust one.

**7.** A sequence satisfies $a_{n+1} = a_n^2 - a_n + 1$ for $n \ge 1$, with $0 < a_1 < 1$. Is $(a_n)$ increasing, decreasing, or could it be either depending on $a_1$?

A) Always increasing (or constant), for any starting value.

B) Always decreasing.

C) Depends on $a_1$.

D) Cannot be determined without more information.

**Answer:** A) Always increasing (or constant), for any starting value.

**Method:** $a_{n+1} - a_n = (a_n^2 - a_n + 1) - a_n = a_n^2 - 2a_n + 1 = (a_n - 1)^2 \ge 0$ always, regardless of $a_n$'s value — so $a_{n+1} \ge a_n$ for every $n$, for *any* real starting value, not just $0 < a_1 < 1$.

> **Investigate further:** The stated restriction $0 < a_1 < 1$ is a red herring: the algebra $a_{n+1} - a_n = (a_n - 1)^2 \ge 0$ holds unconditionally. **So find the one starting value for which the sequence**

> **does *not* strictly increase, and confirm it is the map's fixed point.** Then decide what the sequence does for $a_1 > 1$ — does it still converge?

**8.** For a sequence defined by $a_{n+1} = f(a_n)$ with $f$ continuous, if $(a_n)$ converges to a limit $L$, what must be true about $L$?

A) $f(L) = L$ — $L$ must be a fixed point of $f$.

B) $L = a_1$ always.

C) $f(L) = 0$.

D) No general statement can be made.

**Answer:** A) $f(L) = L$ — $L$ must be a fixed point of $f$.

**Method:** Taking the limit of both sides of $a_{n+1} = f(a_n)$ as $n \to \infty$: the left side $\to L$, and since $f$ is continuous, the right side $\to f(L)$. So $L = f(L)$.

> **Investigate further:** This theorem sits under every fixed-point computation on the sheet. **But note what it does *not* say — construct a sequence $a_{n+1} = f(a_n)$ where $f$ has a perfectly good fixed point yet the sequence never converges.** A3's repelling fixed point is one route in; C1's period-4 map is another.

## Section D Challenge

**1.** A sequence satisfies $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 2}$ (as in C3). Prove rigorously, by induction, that $(a_n)$ is bounded above by 2 for all $n$; then prove $(a_n)$ is strictly increasing; and hence explain why the sequence must converge, identifying its limit.

**Answer:** Bounded above by 2, strictly increasing, converges to 2.

**Method:** *Bounded above by* 2 *(induction):* Base case $a_1 = 1 < 2$. Inductive step: assume $a_k < 2$; then $a_{k+1} = \sqrt{a_k + 2} < \sqrt{2+2} = 2$. By induction, $a_n < 2$ for all $n$.
*Strictly increasing:* Since $0 < a_n < 2$, we have $(a_n - 2) < 0$ and $(a_n + 1) > 0$, so $(a_n - 2)(a_n + 1) < 0$, i.e. $a_n^2 - a_n - 2 < 0$, i.e. $a_n + 2 > a_n^2$. Since $a_n \geq 0$, taking square roots preserves the inequality: $\sqrt{a_n + 2} > a_n$, i.e. $a_{n+1} > a_n$.
*Convergence:* An increasing sequence bounded above must converge (Monotone Convergence Theorem). Taking the limit $L$ of $a_{n+1} = \sqrt{a_n + 2}$: $L = \sqrt{L+2} \Rightarrow L^2 = L + 2 \Rightarrow (L-2)(L+1) = 0 \Rightarrow L = 2$ or $L = -1$. Since every $a_n > 0$, $L \geq 0$, so $L = 2$. □

> **Investigate further:** The rigorous version of C3's numerical observation, combining induction, factoring and the fixed-point theorem. **Now identify which of the three ingredients actually fails if the recurrence is changed to $a_{n+1} = \sqrt{a_n + 6}$ — and find the new limit.** Redo the fixed-point equation first; the structure of the proof should survive unchanged.

**2.** A Möbius map $f(x) = \dfrac{1}{1-x}$ generates a sequence via $a_{n+1} = f(a_n)$. Starting from $a_1 = 2$, find the period of the resulting sequence (the smallest $k > 0$ such that $a_{n+k} = a_n$ for all $n$).

A) 2

B) 3

C) 4

D) 6

**Answer:** B) 3

**Method:** $a_1 = 2$. $a_2 = \dfrac{1}{1-2} = -1$. $a_3 = \dfrac{1}{1-(-1)} = \dfrac{1}{2}$. $a_4 = \dfrac{1}{1-\frac{1}{2}} = 2 = a_1$.

> **Investigate further:** $f(x) = \frac{1}{1-x}$ has order 3 under composition, where C1's map had order 4 — both exact cycles. **Compute $f(f(f(x)))$ symbolically and confirm it simplifies to $x$ for every $x$ in the domain. Then find the two values of $x$ this map cannot be applied to.** Order-3 means the identity is reached in three steps, for *all* valid $x$, not just $a_1 = 2$.

**3.** A sequence is defined by $a_1 = 100$ and $a_{n+1} = \lfloor a_n/2 \rfloor$. Find the smallest $n$ for which $a_n = 0$.

A) 6

B) 7

C) 8

D) 9

**Answer:** C) 8

**Method:** $a_1 = 100, a_2 = 50, a_3 = 25, a_4 = 12, a_5 = 6, a_6 = 3, a_7 = 1, a_8 = 0$.

> **Investigate further:** Floor-halving reaches 0 in about $\log_2(a_1)$ steps: $\log_2 100 \approx 6.6$, and indeed $a_8 = 0$. **Make it exact — find a formula for the number of steps in terms of $a_1$, and identify which starting values are the worst case.** Write $a_1$ in binary; the step count is reading off something very simple about that representation.

**4.** A sequence satisfies $a_1 = 3$ and $a_{n+1} = \dfrac{a_n + \frac{4}{a_n}}{2}$. Which best describes its behaviour?

A) Decreasing (after the first term) and bounded below by 2; converges rapidly to 2.

B) Increasing, converges to 2.

C) Oscillates between values above and below 2.

D) Diverges to infinity.

**Answer:** A) Decreasing (after the first term) and bounded below by 2; converges rapidly to 2.

**Method:** $a_1 = 3$, $a_2 = \dfrac{3 + 4/3}{2} = \dfrac{13/3}{2} = \dfrac{13}{6} \approx 2.167$, $a_3 \approx 2.006$, $a_4 \approx 2.00002$ — rapidly approaching 2 from above. The fixed point solves $x = \dfrac{x + 4/x}{2} \Rightarrow 2x = x + \dfrac{4}{x} \Rightarrow x = \dfrac{4}{x} \Rightarrow x^2 = 4 \Rightarrow x = 2$ (taking the positive root, since all terms stay positive).

> **Investigate further:** This is the Babylonian (Newton) iteration for $\sqrt{4}$; $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$ computes $\sqrt{c}$ for any $c > 0$. **Run it for $c = 2$ from $a_1 = 1$ and count how many decimal places are correct at each step. Then ask: what happens if you start at a negative value?** The fixed-point equation has two roots, and the start decides which one you reach.

**5.** A sequence satisfies $a_{n+1} = a_n(1 - a_n)$ with $a_1 = \frac{1}{2}$. Compute $a_2, a_3$, and determine whether the sequence is monotonic.

A) Decreasing for all $n$, since $a_{n+1} - a_n = -a_n^2 \le 0$ always.

B) Increasing for all $n$.

C) Not monotonic.

D) Constant.

**Answer:** A) Decreasing for all $n$.

**Method:** $a_2 = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$. $a_3 = \frac{1}{4}\left(1 - \frac{1}{4}\right) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$. In general, $a_{n+1} - a_n = a_n(1 - a_n) - a_n = a_n(1 - a_n - 1) = -a_n^2 \leq 0$ always, so the sequence never increases.

> **Investigate further:** C7 in reverse: there $(a_n - 1)^2 \geq 0$ forced the sequence up, here $-a_n^2 \leq 0$ forces it down — writing $a_{n+1} - a_n$ as $\pm(\text{something})^2$ settles monotonicity instantly. **This sequence decreases and is bounded below by 0, so it converges — to what?** Solve $L = L(1 - L)$, then check your answer against the terms you computed.

## Top 5 Patterns — Worksheet #5

1. **Monotonicity via the sign of $a_{n+1} - a_n$.** Often collapses to $\pm(\text{something})^2$, settling the direction instantly without computing terms. *(see A1, A9, B1, C7, D5)*

2. **Fixed points: solve $f(L) = L$.** If a sequence converges, its limit must be a fixed point of the generating function. *(see A3, B4, B5, C8, D4)*

3. **Boundedness + monotonicity together imply convergence.** Neither alone is enough — see Common Trap 1. *(see C3, D1)*

4. **Möbius maps can have small, exact periods under repeated composition.** Check by direct iteration — don't assume a map is aperiodic just because it looks complicated. *(see C1, C2, D2)*

5. **Floor/ceiling notation is just an explicit "round down/up."** Expressions like $\lfloor n/k \rfloor$ often secretly encode periodicity (a remainder) or a counting bound. *(see A5, A10, B7, B8, C5, C6, D3)*

## Common Traps — Worksheet #5

1. **Assuming boundedness alone implies convergence.** $(-1)^n$ is bounded but never converges — monotonicity is also needed. *(see A7, A8)*

2. **Forgetting floor rounds DOWN even for negative numbers.** $\lfloor -2.3 \rfloor = -3$, not $-2$. *(see B7)*

3. **Assuming a sequence must be eventually constant or simply diverging.** Some sequences are genuinely, exactly periodic with period $> 1$ forever. *(see C1, C2, D2)*

4. **Trusting a few computed terms as proof of monotonicity.** Numerical evidence is not a proof — always confirm the sign of $a_{n+1} - a_n$ algebraically for all $n$. *(see D1)*

5. **Assuming a stated restriction on initial conditions is load-bearing.** Some monotonicity results hold far more generally than a question implies — check whether the algebra actually needs the stated constraint. *(see C7)*

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*"Solving $f(L) = L$ tells you where the limit would be, not that there is one. The fixed point is a suspect, not a confession."*

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Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths