Daily Sequences Drill #4 — Answers & Investigations

Sheet by David Akinyele-Aje

Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Recurrence Relations · The Characteristic Equation · Closed-Form Solutions.

Section A Rapid Recognition

1. A recurrence has $a_1 = 2$ and $a_{n+1} = 3a_n - 1$. Find a_2 and a_3.

Answer: $a_2 = 5$, $a_3 = 14$

Method: $a_2 = 3(2) - 1 = 5$. $a_3 = 3(5) - 1 = 14$.

Investigate further: A first-order recurrence with a constant additive term — the shape C1/C2 solve properly later. **Get ahead of them: find the fixed point of $a_{n+1} = 3a_n - 1$, shift by it, and write down a closed form for a_n.** Check it reproduces a_2 and a_3. The fixed point here is not an integer, which is exactly why shifting beats guessing.

2. A recurrence has $a_1 = 1$ and $a_{n+1} = 2a_n + 1$. Find a_4.

Answer: 15

Method: $a_2 = 2(1) + 1 = 3$. $a_3 = 2(3) + 1 = 7$. $a_4 = 2(7) + 1 = 15$.

Investigate further: The pattern $1, 3, 7, 15, \ldots$ is one less than each power of 2, so $a_n = 2^n - 1$. **Now replace the $+1$ by a general constant c: solve $a_{n+1} = 2a_n + c$ in closed form, and find the one starting value for which the sequence never changes.** That value is the fixed point, and it explains the -1 in $2^n - 1$.

3. Write down the characteristic equation of $a_n = 5a_{n-1} - 6a_{n-2}$.

Answer: $x^2 - 5x + 6 = 0$

Method: Replace a_n, a_{n-1}, a_{n-2} with x^2, x^1, x^0 respectively in $a_n = 5a_{n-1} - 6a_{n-2}$.

Investigate further: The substitution is purely formal: linear constant-coefficient recurrences admit solutions $a_n = x^n$, which forces this polynomial exactly. **So does the characteristic equation depend at all on the starting values a_1, a_2?** Take two sequences obeying the same recurrence but starting differently, and say precisely what they share and what they do not.

4. Solve $x^2 - 5x + 6 = 0$ for its roots.

Answer: $x = 2, 3$

Method: $x^2 - 5x + 6 = (x - 2)(x - 3) = 0$.

Investigate further: Vieta again: the roots sum to 5 and multiply to 6, the recurrence's own coefficients. **Use it backwards — build a recurrence whose two roots sum to 0, and describe**

> **what its sequences look like.** With $p = 0$ the recurrence relates a_n only to a_{n-2}; work out what that does to odd and even terms.

5. A recurrence has characteristic roots 2 and 3. Write down the general form of its closed-form solution.

Answer: $a_n = A \cdot 2^n + B \cdot 3^n$

Method: Two distinct real roots r_1, r_2 give a general solution $a_n = Ar_1^n + Br_2^n$, with A, B determined by initial conditions.

> **Investigate further:** The discrete analogue of a second-order ODE's general solution: two independent basis sequences, combined linearly. **For which pairs (A, B) does $a_n = A \cdot 2^n + B \cdot 3^n$ stay bounded as $n \to \infty$?** The answer is far more restrictive than it first looks — consider which term dominates, then what has to happen to the other.

6. The recurrence $a_{n+1} = \frac{1}{2}a_n$, $a_1 = 8$ is secretly which type of sequence from an earlier day? Write its closed form.

Answer: GP with $a_n = 8\left(\frac{1}{2}\right)^{n-1}$

Method: $a_{n+1} = \frac{1}{2}a_n$ is a constant-ratio recurrence, i.e. exactly a geometric sequence with $a = 8, r = \frac{1}{2}$ (Day 3's toolkit).

> **Investigate further:** Recognising this without reaching for a characteristic equation is the point of B9/C6. **This is a GP with $|r| < 1$, so it has a sum to infinity — compute it, and check it against Day 3's A5, which is the identical sequence.** Two days, two toolkits, one sequence: worth confirming they agree exactly.

7. Find the fixed point L of the recurrence $a_{n+1} = 3a_n + 4$ (i.e. solve $L = 3L + 4$).

Answer: $L = -2$

Method: $L = 3L + 4 \Rightarrow -2L = 4 \Rightarrow L = -2$.

> **Investigate further:** L is where the recurrence would sit forever if it ever landed there — the anchor for the whole shift-to-GP trick. **But the trick needs L to exist. For which p does $a_{n+1} = pa_n + c$ have no fixed point at all, and what does the sequence do in that case?** Solve $L = pL + c$ and watch for the division that fails.

8. True or false: every recurrence $a_n = pa_{n-1} + qa_{n-2}$ with real, distinct characteristic roots has a closed form that's a combination of two exponential terms.

Answer: True

Method: This is exactly the general solution form derived in A5: two distinct real roots give a linear combination of two exponential sequences.

> **Investigate further:** This fails the moment the roots repeat or go complex — A10 and C4 handle those. **Why exactly two initial conditions? Explain why one is not enough to pin down A and B, and why a third would generally be inconsistent.** Count unknowns against equations, then say what a third condition would have to satisfy to be compatible.

9. The sequence $a_n = 2^n + 3^n$ satisfies a recurrence $a_n = pa_{n-1} + qa_{n-2}$. Find p and q.

Answer: $p = 5, q = -6$

Method: Since $a_n = 2^n + 3^n$ is built from roots 2 and 3, its characteristic equation is $(x-2)(x-3) = x^2 - 5x + 6 = 0$, i.e. $x^2 = 5x - 6$, giving $a_n = 5a_{n-1} - 6a_{n-2}$.

> **Investigate further:** A3/A4 run backwards: reconstruct the recurrence from the roots rather than solve one. **Now push it up a degree — which recurrence does $a_n = 2^n + 3^n + 5^n$ satisfy?** Build the cubic with roots $2, 3, 5$, expand it, and read the coefficients off; the order of the recurrence always matches the number of distinct roots.

10. A recurrence has a repeated characteristic root $r = 2$. Write down the general form of its closed-form solution.

Answer: $a_n = (A + Bn)2^n$

Method: A repeated root r needs a second, genuinely independent solution alongside r^n — multiplying by n (giving nr^n) supplies it, so the general solution is $a_n = (A + Bn)r^n$.

> **Investigate further:** An extra factor of n keeps the two basis solutions independent. **See why it is forced: try the naive form $a_n = A \cdot 2^n + B \cdot 2^n$ instead and show it collapses to a single constant.** Then explain why that collapse makes it impossible to satisfy two independent initial conditions.

Section B Manipulation Drills

1. Solve the recurrence $a_n = 5a_{n-1} - 6a_{n-2}$ with $a_1 = 1, a_2 = 4$, for the constants A, B in $a_n = A \cdot 2^n + B \cdot 3^n$.

Answer: $A = -\frac{1}{2}, B = \frac{2}{3}$

Method: $n = 1$: $2A + 3B = 1$. $n = 2$: $4A + 9B = 4$. Doubling the first: $4A + 6B = 2$; subtracting from the second: $3B = 2 \Rightarrow B = \frac{2}{3}$. Then $2A + 2 = 1 \Rightarrow A = -\frac{1}{2}$.

> **Investigate further:** Always re-check both conditions: $a_1 = -1 + 2 = 1$ ✓, $a_2 = -2 + 6 = 4$ ✓. **Now the degenerate case: solve the same recurrence with $a_1 = a_2 = 0$. What are A and B, and is the zero sequence the only one with those starting values?** This is the discrete version of "the only solution with zero initial data is zero".

2. For $a_{n+1} = 3a_n + 4$, the fixed point is $L = -2$. Letting $b_n = a_n - L = a_n + 2$, what type of sequence is (b_n)?

A) A GP with ratio 3.

B) An AP with common difference 3.

C) A GP with ratio -2.

D) Neither.

Answer: A) A GP with ratio 3.

Method: $b_{n+1} = a_{n+1} + 2 = (3a_n + 4) + 2 = 3a_n + 6 = 3(a_n + 2) = 3b_n$ — so (b_n) satisfies a pure geometric recurrence.

> **Investigate further:** Subtracting the fixed point cancels the additive constant, leaving pure multiplication. **Test the method's limits: try to run the same shift on $a_{n+1} = a_n + 4$ and**

> **watch it fail.** Identify the exact step that breaks, and say which sequence from Day 1 that recurrence actually generates.

3. Continuing from B2: given $a_1 = 1$ (so $b_1 = 3$), find a_5.

A) 241

B) 243

C) 239

D) 245

Answer: A) 241

Method: $b_1 = a_1 + 2 = 3$, and (b_n) is a GP with ratio 3: $b_n = 3 \cdot 3^{n-1} = 3^n$. So $b_5 = 3^5 = 243$, and $a_5 = b_5 - 2 = 241$.

> **Investigate further:** Convert back from b_n to a_n at the very end — forgetting the shift is the classic slip. **With the closed form in hand, find the smallest n for which a_n exceeds 1000.** Solve the inequality in the power of 3 rather than tabulating, and mind the shift when you compare against 1000.

4. Find the characteristic equation and its roots for $a_n = a_{n-1} + 2a_{n-2}$.

Answer: $x^2 - x - 2 = 0$, roots $x = 2, -1$

Method: $x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 = (x-2)(x+1) \Rightarrow x = 2, -1$.

> **Investigate further:** A negative root is perfectly normal — it contributes a sign-alternating $B(-1)^n$. **Solve this recurrence with $a_1 = a_2 = 1$ and find its closed form.** You should get a single fraction with a 2^n and a $(-1)^n$ in it; check it returns whole numbers at $n = 1, 2, 3$ despite the division.

5. A recurrence has characteristic roots 2 and -1, with $a_1 = 1, a_2 = 5$. Which closed form fits?

A) $a_n = 2^n + (-1)^n$

B) $a_n = 2^n - (-1)^n$

C) $a_n = 2 \cdot 2^n - (-1)^n$

D) $a_n = 2^{n-1} + (-1)^{n-1}$

Answer: A) $a_n = 2^n + (-1)^n$

Method: Check $n = 1$: $2 + (-1) = 1$ ✓. Check $n = 2$: $4 + 1 = 5$ ✓. The other options fail at $n = 1$: B) gives $2 - (-1) = 3$, C) gives $4 + 1 = 5$, D) gives $1 + 1 = 2$ — none match $a_1 = 1$.

> **Investigate further:** Two conditions, two unknowns, so the closed form is unique — always check it against *both* given terms. **Here it comes out as $a_n = 2^n + (-1)^n$. Prove that every term is odd.** A parity argument on the two pieces settles it in one line, with no need to compute any further terms.

6. Which of the following recurrences does NOT have a characteristic equation with two distinct real roots?

A) $a_n = 5a_{n-1} - 6a_{n-2}$

B) $a_n = 4a_{n-1} - 4a_{n-2}$

C) $a_n = a_{n-1} + 2a_{n-2}$

D) $a_n = 3a_{n-1} - 2a_{n-2}$

Answer: B) $a_n = 4a_{n-1} - 4a_{n-2}$

Method: Its characteristic equation is $x^2 - 4x + 4 = (x-2)^2 = 0$ — a repeated root, discriminant $16 - 16 = 0$. The other three all have positive discriminant (A: $25 - 24 = 1$; C: $1 + 8 = 9$; D: $9 - 8 = 1$), giving distinct real roots.

> **Investigate further:** The discriminant $p^2 + 4q$ decides it faster than factorising each option. **Use it to answer a general question: for which q does $a_n = a_{n-1} + qa_{n-2}$ have two distinct real roots?** You should get a single inequality with a fractional boundary — and it is worth checking what happens exactly at that boundary.

7. A recurrence has repeated characteristic root $r = 3$, with $a_1 = 6, a_2 = 27$. Using $a_n = (A+Bn)3^n$, find A and B.

A) $A = 1, B = 1$

B) $A = 2, B = 1$

C) $A = 1, B = 2$

D) $A = 0, B = 2$

Answer: A) $A = 1, B = 1$

Method: $(A + B \cdot 1) \cdot 3^1 = 6 \Rightarrow A + B = 2$. $(A + B \cdot 2) \cdot 3^2 = 27 \Rightarrow A + 2B = 3$. Subtracting: $B = 1$, then $A = 1$.

> **Investigate further:** Double-check: $a_1 = 2 \cdot 3 = 6$ ✓, $a_2 = 3 \cdot 9 = 27$ ✓, so $a_n = (n+1)3^n$. **Compute the ratio a_{n+1}/a_n as a formula in n. Does it ever equal 3 exactly, and what does it approach?** The gap between "tends to 3" and "equals 3" is exactly what the repeated root introduces.

8. Using the closed form from B7, find a_4.

Answer: 405

Method: $a_n = (1+n)3^n$. $a_4 = (1+4) \cdot 3^4 = 5 \times 81 = 405$.

> **Investigate further:** The extra factor of n makes this outgrow any plain $c \cdot 3^n$. **Show that decisively: form the ratio $\frac{(n+1)3^n}{c \cdot 3^n}$ and explain why no constant c can keep pace as n grows.** Then note $a_4 = 405$ happens to equal $5 \cdot 3^4$ — a coincidence at one value of n, not a general match.

9. A recurrence is defined by $a_{n+1} = a_n + 5$, $a_1 = 2$. Is this recurrence's closed form better found using the AP toolkit (Day 1) or the characteristic-equation toolkit (Day 4)?

A) The AP toolkit — it's simply $a_n = 2 + 5(n-1)$.

B) The characteristic-equation toolkit — solve $x = x + 5$.

C) Both give the same closed form, but the characteristic equation is faster.

D) Neither toolkit applies.

Answer: A) The AP toolkit — it's simply $a_n = 2 + 5(n-1)$.

Method: A constant additive step (+5 every time, no multiplication) is precisely Day 1's arithmetic sequence definition — the characteristic-equation machinery is unnecessary (and its "fixed point" equation $x = x + 5$ has no solution, signalling this recurrence sits outside that method's normal use case).

> **Investigate further:** Knowing when a lighter toolkit suffices matters as much as knowing the heavy one. **Solve this both ways — once with Day 1's AP formula, once by treating it as a recurrence with root 1 and a constant forcing term — and confirm they agree.** The second route should show you why the answer is linear in n rather than exponential.

10. A recurrence has characteristic equation $x^2 - 6x + 9 = 0$. What is special about its roots?

A) Two distinct real roots.

B) A repeated real root.

C) Two complex roots.

D) No real roots.

Answer: B) A repeated real root.

Method: $x^2 - 6x + 9 = (x-3)^2 = 0$ — discriminant $36 - 36 = 0$.

> **Investigate further:** Same structure as B7's $a_n = 6a_{n-1} - 9a_{n-2}$. **Confirm it: expand $(x-3)^2$ and check it really is $x^2 - 6x + 9$, then state the general rule for reading a repeated root straight off $x^2 + px + q$.** The condition is a statement about the discriminant — write it as an equation in p and q.

Section C Substitution & Structure

1. Which substitution transforms $a_{n+1} = 4a_n - 3$ into a pure geometric recurrence $b_{n+1} = 4b_n$?

A) $b_n = a_n - 1$

B) $b_n = a_n + 1$

C) $b_n = a_n - 3$

D) $b_n = a_n/4$

Answer: A) $b_n = a_n - 1$

Method: Fixed point: $L = 4L - 3 \Rightarrow -3L = -3 \Rightarrow L = 1$. So $b_n = a_n - 1$, giving $b_{n+1} = a_{n+1} - 1 = (4a_n - 3) - 1 = 4a_n - 4 = 4(a_n - 1) = 4b_n$.

> **Investigate further:** The fixed point is always $L = \frac{c}{1-p}$ for $p \neq 1$ — worth having on recall. **For the family $a_{n+1} = 4a_n - c$, exactly which c give an integer fixed point?** Write L in terms of c and impose integrality; the answer is a divisibility condition, not an inequality.

2. The recurrence $a_{n+1} = 4a_n - 3$ with $a_1 = 2$ has closed form $a_n = 1 + c \cdot 4^{n-1}$. Find c.

A) 1

B) 2

C) 3

D) 4

Answer: A) 1

Method: At $n = 1$: $a_1 = 1 + c \cdot 4^0 = 1 + c = 2 \Rightarrow c = 1$.

> **Investigate further:** This c is exactly b_1 from the shift-to-GP method — the two approaches are one technique in two notations. **So which starting value a_1 makes this sequence constant, and what happens to the sequence when a_1 is below that value?** Describe the behaviour in both directions from the fixed point.

3. A recurrence has characteristic roots $r_1 = 5$ and $r_2 = -2$. Using $p = r_1 + r_2$ and $q = -r_1 r_2$, write down the recurrence $a_n = pa_{n-1} + qa_{n-2}$.

A) $a_n = 3a_{n-1} + 10a_{n-2}$

B) $a_n = 3a_{n-1} - 10a_{n-2}$

C) $a_n = -3a_{n-1} + 10a_{n-2}$

D) $a_n = 7a_{n-1} - 10a_{n-2}$

Answer: A) $a_n = 3a_{n-1} + 10a_{n-2}$

Method: $p = 5 + (-2) = 3$. $q = -(5)(-2) = 10$. Check via the characteristic equation directly: $(x - 5)(x + 2) = x^2 - 3x - 10 = 0 \Rightarrow x^2 = 3x + 10 \Rightarrow a_n = 3a_{n-1} + 10a_{n-2}$.

> **Investigate further:** Rebuild the characteristic equation from its roots rather than trusting $p = r_1 + r_2$, $q = -r_1 r_2$ from memory. **Do exactly that here: expand $(x - 5)(x + 2)$ and confirm it matches the recurrence you wrote.** Then say why the sign of q flips relative to the product of the roots — that sign is the most common error in this direction.

4. A recurrence has characteristic equation $x^2 - 2x + 5 = 0$. What can be said about its roots?

A) Two distinct real roots.

B) A repeated real root.

C) Two complex conjugate roots, $1 \pm 2i$.

D) No roots exist.

Answer: C) Two complex conjugate roots, $1 \pm 2i$.

Method: Discriminant $= (-2)^2 - 4(1)(5) = 4 - 20 = -16 < 0$, so the roots are complex: $x = \dfrac{2 \pm \sqrt{-16}}{2} = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$.

> **Investigate further:** A negative discriminant does not mean "no solution" — the recurrence still has a well-defined, real, oscillating closed form. **Find the two roots explicitly and compute their modulus. Does this sequence's magnitude grow, shrink, or hold steady?** The modulus, not the real part, is what decides it — compare with Day 3's $|r| < 1$.

5. Which technique is most appropriate for solving $a_{n+1} = 2a_n + 3^n$ (note the non-constant additive term)?

A) The constant-shift trick ($b_n = a_n - L$ for a fixed L) — works directly.

B) The homogeneous characteristic-equation method, plus a particular solution of the form $C \cdot 3^n$ added to the general solution.

C) Treating it as a plain AP.

D) It cannot be solved by any method covered so far.

Answer: B) The homogeneous method plus a particular solution $C \cdot 3^n$.

Method: The constant-shift trick only cancels a *constant* forcing term — here the forcing term 3^n grows with n, so no fixed constant L can absorb it. The standard fix is to add a particular solution of the same form as the forcing term ($C \cdot 3^n$) to the homogeneous general solution.

Investigate further: The discrete analogue of "guess a particular solution matching the forcing term". **Now break it deliberately: what goes wrong if the forcing term is 2^n instead of 3^n, given that 2 is already a characteristic root?** Try the obvious guess $C \cdot 2^n$, watch it fail, and work out what extra factor rescues it.

6. Which of the following recurrences is secretly equivalent to an arithmetic sequence — Day 1's toolkit is sufficient, no characteristic equation needed?

A) $a_{n+1} = a_n + 7$

B) $a_{n+1} = 2a_n$

C) $a_{n+1} = 2a_n + 7$

D) $a_n = 5a_{n-1} - 6a_{n-2}$

Answer: A) $a_{n+1} = a_n + 7$

Method: A) is a constant additive step — an AP, Day 1's toolkit exactly. B) is a constant-ratio step — a GP (Day 3). C) needs the shift-to-GP trick (still not a plain AP). D) genuinely needs the full characteristic equation.

Investigate further: Triage in miniature: ask whether Day 1 or Day 3 already covers it before reaching for the heavy machinery. **Make the test precise — for exactly which (p, q) does $a_n = pa_{n-1} + qa_{n-2}$ reduce to an arithmetic sequence?** Work out what both roots must be, then convert that back into a single pair of coefficients.

7. A recurrence $a_n = pa_{n-1} + qa_{n-2}$ has BOTH characteristic roots equal to 1. What does this imply about a_n as $n \to \infty$ (assuming (a_n) is not identically zero)?

A) Exponential growth.

B) Linear growth in n (or constant) — effectively an AP.

C) The sequence must be periodic.

D) The sequence must converge to a finite limit.

Answer: B) Linear growth in n (or constant) — effectively an AP.

Method: A repeated root $r = 1$ gives closed form $a_n = (A + Bn)(1)^n = A + Bn$ — exactly linear in n (an AP) if $B \neq 0$, or constant if $B = 0$.

Investigate further: The two-exponential structure collapses to Day 1's AP exactly when both roots equal 1. **Write out the closed form in that case and identify the AP's common difference in terms of A and B.** Then answer the natural follow-up: which choice of A, B gives a

> constant sequence rather than a genuinely increasing one?

8. A recurrence has characteristic roots r and $\frac{1}{r}$ (reciprocal roots, $r \neq \pm 1$). What can be said about p and q in $a_n = pa_{n-1} + qa_{n-2}$?

A) $q = -1$ always, while $p = r + \frac{1}{r}$ can be any value with $|p| \geq 2$.

B) $p = -1$ always.

C) p and q must both equal 1.

D) No general statement can be made.

Answer: A) $q = -1$ always, $p = r + \frac{1}{r}$ can be any value with $|p| \geq 2$.

Method: $q = -r_1 r_2 = -r \cdot \frac{1}{r} = -1$, independent of r. And $p = r + \frac{1}{r}$, which by AM–GM-style reasoning satisfies $\left|r + \frac{1}{r}\right| \geq 2$ for any nonzero real r (with equality only at $r = \pm 1$).

> **Investigate further:** The bound $\left|r + \frac{1}{r}\right| \geq 2$ is an Algebra-pillar tool resurfacing to describe reciprocal roots. **Given that $p = r + \frac{1}{r}$, for which *integer* values of p is r itself rational?** Solve the quadratic for r and ask when its discriminant is a perfect square — the surviving cases are precisely the ones the question already excludes.

Section D Challenge

1. A sequence is defined by $a_1 = 1, a_2 = 2$, and for $n \geq 3$: $a_n = \dfrac{a_{n-1} + a_{n-2}}{2}$. Find $\lim_{n\to\infty} a_n$.

Answer: $\dfrac{5}{3}$

Method: Characteristic equation: $2x^2 = x + 1 \Rightarrow 2x^2 - x - 1 = 0 = (2x+1)(x-1) \Rightarrow x = 1, -\frac{1}{2}$. General solution: $a_n = A + B\left(-\frac{1}{2}\right)^n$. Using $a_1 = 1$: $A - \frac{B}{2} = 1$. Using $a_2 = 2$: $A + \frac{B}{4} = 2$. Subtracting: $\frac{3B}{4} = 1 \Rightarrow B = \frac{4}{3}$, then $A = 1 + \frac{2}{3} = \frac{5}{3}$. As $n \to \infty$, $\left(-\frac{1}{2}\right)^n \to 0$, so $a_n \to A = \dfrac{5}{3}$.

> **Investigate further:** The terms $1, 2, 1.5, 1.75, 1.625, \ldots$ oscillate in on $\frac{5}{3}$; a root of exactly 1 alongside a decaying one gives a finite non-zero limit. **Find the general formula for that limit in terms of a_1 and a_2, then test it on $a_1 = 0, a_2 = 3$.** Solve the characteristic equation first — the limit is the coefficient of the root-1 term, and it is a weighted average, not a plain one.

2. A recurrence satisfies $a_{n+1} = 6a_n - 9a_{n-1}$ (for $n \geq 2$), with $a_1 = 1, a_2 = 6$. Using the repeated-root closed form $a_n = (A + Bn)3^n$, find a_5.

A) 405

B) 243

C) 324

D) 486

Answer: A) 405

Method: Characteristic equation $x^2 - 6x + 9 = (x-3)^2 = 0$ — repeated root 3. Solving $a_n = (A+Bn)3^n$: $n = 1$: $(A+B)(3) = 1 \Rightarrow A + B = \frac{1}{3}$. $n = 2$: $(A+2B)(9) = 6 \Rightarrow A + 2B = \frac{2}{3}$. Subtracting: $B = \frac{1}{3}$, so $A = 0$. Thus $a_n = \frac{n}{3} \cdot 3^n = n \cdot 3^{n-1}$. $a_5 = 5 \times 3^4 = 5 \times 81 = 405$.

> **Investigate further:** $a_n = n \cdot 3^{n-1}$ gives $1, 6, 27, 108, 405, \ldots$, and $a_3 = 6(6) - 9(1) = 27$ ✓. **Compare this with B7, which had the same repeated root 3 but closed form $(n+1)3^n$. Why do two sequences with identical characteristic equations look so different?** The answer is entirely in the initial conditions — state precisely what the characteristic equation does and does not determine.

3. A recurrence has characteristic equation $x^2 + 4 = 0$ (i.e. $a_n = -4a_{n-2}$). What is true about its behaviour?

A) The sequence oscillates in sign while its magnitude grows like 2^n — consistent with complex roots of magnitude 2.

B) The sequence converges to 0.

C) The sequence is eventually constant.

D) The sequence is purely real-valued growth with no oscillation.

Answer: A) The sequence oscillates in sign while its magnitude grows like 2^n.

Method: $x^2 = -4 \Rightarrow x = \pm 2i$, purely imaginary roots of magnitude 2. Testing directly with $a_1 = 1, a_2 = 0$: $a_3 = -4a_1 = -4$, $a_4 = -4a_2 = 0$, $a_5 = -4a_3 = 16$, $a_6 = -4a_4 = 0$, $a_7 = -4a_5 = -64$ — the nonzero terms alternate sign and grow by a factor of 4 every two steps (i.e. magnitude 2 per step), exactly matching complex roots of modulus 2.

> **Investigate further:** For complex roots $re^{i\theta}$ the magnitude grows like r^n while the sign cycles with θ; purely imaginary roots give a full cycle every 4 steps. **Verify the period directly: using $a_n = -4a_{n-2}$, compute a_9 in terms of a_1.** Apply the recurrence four times and watch both the sign pattern and the factor of 4 per step accumulate.

4. For which of the following (p, q) pairs does $a_n = pa_{n-1} + qa_{n-2}$ produce a closed form that reduces exactly to Day 1's arithmetic-sequence form (i.e. a_n linear in n), for suitable initial conditions?

A) $(p, q) = (2, -1)$

B) $(p, q) = (1, 1)$

C) $(p, q) = (-2, 1)$

D) $(p, q) = (3, -2)$

Answer: A) $(p, q) = (2, -1)$

Method: Linear-in-n behaviour needs a repeated root of exactly 1: $(x-1)^2 = x^2 - 2x + 1 = 0$, giving $p = 2, q = -1$ (matching $x^2 - px - q = 0$). Checking D: $(p, q) = (3, -2)$ gives $x^2 - 3x + 2 = (x-1)(x-2) = 0$ — roots $1, 2$, distinct, so the general solution $A + B \cdot 2^n$ is linear in n only in the degenerate special case $B = 0$, not for suitable general initial conditions.

> **Investigate further:** C7 read backwards: from the desired linear growth, deduce the characteristic equation and hence (p, q). **Having found the pair, check the edge case — what does that recurrence produce when $a_1 = a_2$?** An AP with a repeated root at 1 still has one degree of freedom left, and this is where it shows.

5. A recurrence has characteristic roots r_1, r_2 with $|r_1| < 1$ and $|r_2| < 1$. What happens to a_n as $n \to \infty$, regardless of initial conditions?

A) $a_n \to 0$ always.

B) a_n diverges to infinity.

C) a_n oscillates forever without settling.

D) It depends on the initial conditions.

Answer: A) $a_n \to 0$ always.

Method: $a_n = Ar_1^n + Br_2^n$ (or the repeated-root analogue). Since $|r_1| < 1$ and $|r_2| < 1$, both $r_1^n \to 0$ and $r_2^n \to 0$ as $n \to \infty$, regardless of the constants A, B (which are fixed by the initial conditions but don't affect whether the limit is 0).

> **Investigate further:** The recurrence version of Day 3's $|r| < 1$: *both* roots must lie inside the unit circle, not just one. **So what happens in the boundary case where $|r_1| < 1$ but $|r_2| = 1$ exactly?** Decide whether the sequence still converges, merely stays bounded, or neither — and note that $r_2 = 1$ and $r_2 = -1$ behave differently.

Top 5 Patterns — Worksheet #4

1. **Characteristic equation: turn $a_n = pa_{n-1} + qa_{n-2}$ into $x^2 - px - q = 0$.** The entire toolkit starts here. *(see A3, A4, A9, B4, C3)*

2. **Distinct real roots $\to a_n = Ar_1^n + Br_2^n$; repeated root $\to a_n = (A + Bn)r^n$.** Always check the discriminant first to know which shape applies. *(see A5, A10, B5, B7, C7)*

3. **First-order recurrences $a_{n+1} = pa_n + c$ shift to a pure GP via the fixed point $L = \frac{c}{1-p}$.** Let $b_n = a_n - L$, solve the GP, shift back. *(see A7, B2, B3, C1, C2)*

4. **Recognise the degenerate cases first.** Constant-difference recurrences are Day 1's AP; constant-ratio recurrences are Day 3's GP — don't reach for the characteristic equation when a lighter toolkit already solves it. *(see A6, B9, C6)*

5. **Complex characteristic roots signal oscillation with exponentially changing magnitude**, not simple convergence or divergence. *(see C4, D3)*

Common Traps — Worksheet #4

1. **Forgetting to check for a repeated root before using the two-distinct-roots closed form.** A zero discriminant needs $(A + Bn)r^n$ instead. *(see B6, B10)*

2. **Mixing up the sign convention when reading p, q off from given roots.** Remember $x^2 - px - q = 0$, not $x^2 + px + q = 0$. *(see C3, D4)*

3. **Applying the constant-shift trick to a recurrence with a non-constant forcing term.** $b_n = a_n - L$ only cancels a fixed constant — it does nothing against a term like 3^n. *(see C5)*

4. **Assuming every recurrence converges to a finite limit.** Convergence needs *every* characteristic root to have magnitude strictly less than 1. *(see D5)*

5. **Solving for A, B using only one initial condition.** A second-order recurrence's closed form has two free constants and needs two equations to pin them both down. *(see B1, B5, B7)*

"A repeated root does not give you $Ar^n + Br^n$. It gives you $(A + Bn)r^n$, and the n is not decoration."

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths