Daily Sequences Drill #3 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Geometric Sequences & Series · Sum to Infinity · The Convergence Condition $|r| < 1$.

Section A Rapid Recognition

1. Write down the common ratio of the GP $3, 6, 12, 24, \ldots$

Answer: 2

Method: $6/3 = 2$, $12/6 = 2$, $24/12 = 2$ — constant ratio 2.

> **Investigate further:** A GP's ratio is *next* $\div$ *current*, never the reverse. **So what single equation must three numbers** a, b, c **satisfy to sit consecutively in a GP?** Derive it from the two ratios being equal, check it on $3, 6, 12$, and note that it makes no reference to the ratio itself.

2. A GP has first term 5 and common ratio 2. Find the 6th term.

Answer: 160

Method: $a_6 = 5 \times 2^5 = 5 \times 32 = 160$.

> **Investigate further:** Compare $a + (n-1)d$ with ar^{n-1}: multiplication replaces addition throughout. **For** $a > 0$**, exactly which** r **make the GP strictly increasing? Now answer the same question for** $a < 0$**.** The two answers are different, and the sign of a is the whole reason.

3. A GP has first term 2 and common ratio 3. Find S_5.

Answer: 242

Method: $S_5 = \dfrac{2(3^5 - 1)}{3 - 1} = \dfrac{2 \times 242}{2} = 242$.

> **Investigate further:** Check directly: $2 + 6 + 18 + 54 + 162 = 242$ ✓. **Notice** $242 = 3^5 - 1$**. Prove that for this GP** $S_n = 3^n - 1$ **exactly, for every** n**.** Substitute $a = 2, r = 3$ into the sum formula and watch the 2 cancel — then say what it is about a and r here that makes the answer this clean.

4. State the condition on r for an infinite GP's sum to converge.

Answer: $|r| < 1$

Method: The infinite sum $a + ar + ar^2 + \cdots$ only settles to a finite value when each term shrinks toward 0, which needs $|r| < 1$.

> **Investigate further:** The one condition the sheet is built around. **Test the boundary yourself: with** $a = 1$ **and** $r = -1$**, write out the first six partial sums.** Do they approach a limit, oscillate, or grow? Then explain why $|r| < 1$ has to be a *strict* inequality on the magnitude, not on r itself.

5. A GP has first term 8 and common ratio $\frac{1}{2}$. Find its sum to infinity.

Answer: 16

Method: $S_\infty = \dfrac{8}{1-\frac{1}{2}} = \dfrac{8}{\frac{1}{2}} = 16.$

> **Investigate further:** Partial sums $8, 12, 14, 15, 15.5, \ldots$ climb toward 16 without reaching it. The shortfall after n terms is exactly $16 - S_n$. **Find a formula for that shortfall, then determine how many terms are needed to get within 0.01 of 16.** No logarithms required — the shortfall is a power of 2.

6. Which of these three sequences is geometric: (i) $2, 4, 6, 8$ (ii) $2, 4, 8, 16$ (iii) $2, 4, 7, 11$?

Answer: Sequence (ii)

Method: (i) has constant difference 2 — arithmetic, not geometric. (iii) has differences $2, 3, 4$ and ratios $2, 1.75, 1.57\ldots$ — neither.

> **Investigate further:** Test against *both* definitions before concluding "neither" — (iii) genuinely fits no pattern. **Can a single sequence be both arithmetic and geometric at once?** Write down what each definition demands, solve the two conditions together, and describe every sequence that satisfies both.

7. Write down the sum-to-infinity formula for a GP with first term a and ratio r (where $|r| < 1$).

Answer: $\dfrac{a}{1-r}$

Method: Take the finite-sum formula $S_n = \dfrac{a(1-r^n)}{1-r}$ and let $n \to \infty$: since $|r| < 1$, $r^n \to 0$, leaving $S_\infty = \dfrac{a}{1-r}$.

> **Investigate further:** Taking a limit of the finite formula is exactly how D1's proof works. **Derive $S_\infty = \frac{a}{1-r}$ a second way, without any limit: write out S, multiply it by r, and subtract.** Then identify the exact step in that argument that silently assumes $|r| < 1$ — it is easy to miss.

8. True or false: a GP with common ratio $r = 1$ has a well-defined sum to infinity.

Answer: False

Method: At $r = 1$ every term equals a, so the sum after n terms is na, which grows without bound (unless $a = 0$) — it never settles to a finite value.

> **Investigate further:** $r = 1$ is precisely the excluded boundary. **Put $r = 1$ into $\frac{a(1-r^n)}{1-r}$ and see what breaks; then work out what S_n actually is for a GP with $r = 1$.** The true answer is obvious from the sequence itself, which makes it a good check on whether a formula is being applied blindly.

9. A GP has first term 100 and common ratio $-\frac{1}{2}$. Find its sum to infinity.

Answer: $\dfrac{200}{3}$

Method: $S_\infty = \dfrac{100}{1-(-\frac{1}{2})} = \dfrac{100}{\frac{3}{2}} = \dfrac{200}{3}.$

> **Investigate further:** A negative ratio *increases* the denominator: $1-(-\frac{1}{2})=\frac{3}{2}$. **Compare this GP's sum with the same first term but $r=+\frac{1}{2}$. What is the ratio of the two sums, and does that factor depend on a at all?** Then generalise: how do S_∞ for r and for $-r$ compare in general?

10. Write the next term of the GP $81, 27, 9, 3, \ldots$

Answer: 1

Method: Ratio is $\frac{1}{3}$: $81, 27, 9, 3, 1, \ldots$

> **Investigate further:** The terms shrink toward 0 without reaching it — what makes a sum to infinity possible. **But is "terms tend to 0" *enough* to guarantee a finite sum? Test it on $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$, whose terms certainly tend to 0.** Group the terms in blocks of $2, 4, 8, \ldots$ and bound each block from below.

Section B Manipulation Drills

1. A GP has first term 4 and common ratio 3. Find the smallest n for which $S_n > 1000$.

Answer: $n = 6$

Method: $S_n = \dfrac{4(3^n-1)}{2} = 2(3^n-1) > 1000 \Rightarrow 3^n > 501$. $3^5 = 243$ (too small), $3^6 = 729 > 501$. So $n = 6$.

> **Investigate further:** Test boundary powers directly rather than reaching for logarithms under time pressure. **Here $S_n = 2(3^n-1)$, so the condition reduces to a single inequality in 3^n — write it down and confirm which power of 3 first clears it.** Then state the general shortcut: for $S_n > T$, what inequality in r^n are you always really solving?

2. For which values of r does the infinite GP with first term 5 have a finite sum?

A) all real r

B) $r > 0$ only

C) $-1 < r < 1$

D) $r \neq 1$

Answer: C) $-1 < r < 1$

Method: The sum-to-infinity formula $S_\infty = \dfrac{5}{1-r}$ is only valid, i.e. only converges, when $|r| < 1$ — the first term's value (5) plays no role in whether convergence happens.

> **Investigate further:** Convergence depends *only* on r, never on a. **So does a influence anything about the infinite sum at all — and what happens in the single special case $a = 0$?** Say precisely which features of the series a controls and which it does not.

3. A GP has first term 3 and common ratio $\frac{2}{3}$. Find its sum to infinity.

Answer: 9

Method: $S_\infty = \dfrac{3}{1-\frac{2}{3}} = \dfrac{3}{\frac{1}{3}} = 9$.

> **Investigate further:** The terms $3, 2, \frac{4}{3}, \frac{8}{9}, \ldots$ have running total climbing toward 9. **How many terms are needed before the partial sum passes half the total, i.e. exceeds** 4.5**?** Set up the inequality in $\left(\frac{2}{3}\right)^n$ — the answer is far smaller than most people guess, which is itself worth noticing.

4. A GP has sum to infinity 20 and first term 5. Find r.

A) $\frac{3}{4}$

B) $\frac{1}{4}$

C) $\frac{4}{5}$

D) $-\frac{3}{4}$

Answer: A) $\frac{3}{4}$

Method: $20 = \dfrac{5}{1-r} \Rightarrow 1 - r = \dfrac{5}{20} = \dfrac{1}{4} \Rightarrow r = \dfrac{3}{4}$.

> **Investigate further:** Isolate $(1-r)$ by cross-multiplying, then solve for r last — wrong order invites sign slips. **Now run it backwards: for a GP with $a = 5$, which sums to infinity are actually achievable as r ranges over $|r| < 1$?** You should find the reachable values are not all of $\mathbb{R}$ — describe exactly which are excluded, and why S_∞ can never be small.

5. A GP has first term 1 and sum to infinity 2. Find the second term.

A) $\frac{1}{2}$

B) $\frac{1}{4}$

C) 2

D) 1

Answer: A) $\frac{1}{2}$

Method: $2 = \dfrac{1}{1-r} \Rightarrow 1 - r = \dfrac{1}{2} \Rightarrow r = \dfrac{1}{2}$. Second term $= 1 \times \dfrac{1}{2} = \dfrac{1}{2}$.

> **Investigate further:** The question asks for the second *term*, not the ratio — here they coincide numerically, which is exactly the trap. **Change the first term to $a = 3$ with the same sum to infinity 2: does the coincidence survive?** Then say for which a the second term equals the common ratio.

6. An AP and a GP both have first term 4. The AP has common difference 2; the GP has common ratio 2. Which sequence has the larger 10th term?

A) The AP

B) The GP

C) They are equal

D) Cannot be determined

Answer: B) The GP

Method: AP 10th term: $4 + 9(2) = 22$. GP 10th term: $4 \times 2^9 = 4 \times 512 = 2048$. $2048 \gg 22$.

> **Investigate further:** Geometric eventually beats arithmetic — by the 10th term the gap is already near two orders of magnitude. **But they start equal at $n = 1$. From which term onward is the GP permanently ahead, and can you prove it never falls behind again?** Induction on the ratio between consecutive terms is the clean route.

7. A GP has all terms positive and common ratio $r > 1$. Does its sum to infinity exist?

A) Yes, always.

B) No, never — it diverges.

C) Only if the first term is small.

D) Only if $r < 2$.

Answer: B) No, never — it diverges.

Method: $r > 1$ means $|r| > 1$, which is exactly the case excluded from convergence — each term is strictly bigger than the last, so the partial sums grow without bound.

> **Investigate further:** "All terms positive" is a red herring: only $|r|$ decides convergence. **Build the sharpest counterexamples you can — a GP with all terms positive that diverges, and one with terms of alternating sign that converges.** Then state what, if anything, the sign of the terms does tell you about the sum.

8. Express the recurring decimal $0.\overline{3} = 0.333\ldots$ as a fraction, using the geometric series sum-to-infinity formula.

Answer: $\frac{1}{3}$

Method: $0.\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$, a GP with $a = \frac{3}{10}$, $r = \frac{1}{10}$. $S_\infty = \frac{3/10}{1 - 1/10} = \frac{3/10}{9/10} = \frac{3}{9} = \frac{1}{3}$.

> **Investigate further:** Every recurring decimal is a GP with $a =$ (repeating block over its place value) and $r = 10^{-(\text{block length})}$. **Use it on $0.\overline{142857}$ and see which famous fraction appears. Then answer the structural question: why must every recurring decimal be rational?** The block length controls the denominator.

9. A GP has first term 2 and common ratio -3. Find S_4.

A) 40

B) -40

C) 160

D) -160

Answer: B) -40

Method: $S_4 = \frac{2((-3)^4 - 1)}{-3 - 1} = \frac{2(81 - 1)}{-4} = \frac{160}{-4} = -40.$

> **Investigate further:** The trap is mishandling the signed denominator $1 - r = 4$. **Compute $S_1, \ldots, S_5$ for this GP and describe the pattern the partial sums follow.** With $r = -3$ they neither increase nor converge — say exactly what they do, and connect that to where $|r| = 3$ sits relative to 1.

10. Recall from Day 2 that $(1-x)^{-1} = 1 + x + x^2 + \cdots$ for $|x| < 1$. Using $x = \frac{1}{3}$, what does this series sum to?

A) $\frac{3}{2}$

B) $\frac{2}{3}$

C) 3

D) 2

Answer: A) $\frac{3}{2}$

Method: $(1-x)^{-1} = \dfrac{1}{1-x}$. At $x = \dfrac{1}{3}$: $\dfrac{1}{1-1/3} = \dfrac{1}{2/3} = \dfrac{3}{2}$.

> **Investigate further:** $(1-x)^{-1}$ evaluated at a specific x *is* a sum-to-infinity calculation — this is B3's GP in Day 2's notation. **So what is the exact range of x for which that series identity is valid, and what is it about $x = 1$ that makes both sides fail simultaneously?** Compare the two sides just inside and just outside the range.

Section C Substitution & Structure

1. A GP has first term 1 and unknown common ratio r. Its sum to infinity exists and equals 4. What must be true about r?

A) $r = \frac{3}{4}$ exactly.

B) $-1 < r < 1$, but no exact value is determinable.

C) $r = \frac{4}{3}$.

D) Infinitely many values of r satisfy this.

Answer: A) $r = \frac{3}{4}$ exactly

Method: $\dfrac{1}{1-r} = 4 \Rightarrow 1 - r = \dfrac{1}{4} \Rightarrow r = \dfrac{3}{4}$ — a single equation in one unknown has exactly one solution.

> **Investigate further:** With a fixed, $S_\infty = 4$ pins r down uniquely — nothing is underdetermined. **Now let the target sum vary: for $a = 1$, which sums to infinity are achievable, and which are impossible no matter how r is chosen?** Solve $S = \frac{1}{1-r}$ for r and impose $|r| < 1$ — the impossible band is larger than it first appears.

2. A geometric sequence has first term a and common ratio r, both positive integers with $r > 1$. Let S_n denote the sum of the first n terms. Given $S_{18} - S_{12} = kS_6$ for some positive integer k, find the smallest possible value of k. *(after TMUA 2022 Paper 1 Q8)*

A) 2^6

B) 2^{12}

C) 2^{18}

D) $\dfrac{2^6}{2^6 - 1}$

E) $2^6(2^6 - 1)$

Answer: B) 2^{12}

Method: $S_{18} - S_{12} = \dfrac{a(r^{18} - r^{12})}{r-1} = \dfrac{ar^{12}(r^6 - 1)}{r-1} = r^{12} \cdot \dfrac{a(r^6-1)}{r-1} = r^{12} S_6$. So $k = r^{12}$, minimised over integers $r > 1$ at $r = 2$: $k = 2^{12} = 4096$.

> **Investigate further:** The exponent is always the index gap between the two subtracted sums. **Prove the general rule: show $S_{m+j} - S_m = r^m S_j$ for any GP, then use it to predict k before computing anything. Check your rule against the real TMUA question (2022 Paper 1 Q8)**, which uses $S_{30} - S_{20} = kS_{10}$ and minimises k to 2^{20}.

3. An infinite geometric series has first term $a = 6$ and common ratio $r = \dfrac{3k}{10}$, where k is chosen uniformly at random from the integers $-5, -4, \ldots, 4, 5$ (11 equally likely values). Find the probability that the series converges *and* its sum to infinity exceeds 5. *(after TMUA 2023 Paper 1 Q18)*

A) $\frac{4}{11}$

B) $\frac{5}{11}$

C) $\frac{7}{11}$

D) $\frac{3}{11}$

Answer: A) $\frac{4}{11}$

Method: Convergence needs $|r| < 1$: $\left|\frac{3k}{10}\right| < 1 \Rightarrow |k| < \frac{10}{3} \approx 3.33 \Rightarrow k \in \{-3, \ldots, 3\}$ (7 of the 11 values). Among all 11 possible k, $S_\infty = \dfrac{6}{1 - 3k/10} = \dfrac{60}{10 - 3k} > 5$ requires $60 > 5(10 - 3k) \Rightarrow 60 > 50 - 15k \Rightarrow 15k > -10 \Rightarrow k > -\frac{2}{3} \Rightarrow k \geq 0$ (since $10 - 3k > 0$ throughout the converging range, the inequality direction is preserved). Combined with convergence, valid $k \in \{0, 1, 2, 3\}$ — 4 values. Probability $= \dfrac{4}{11}$.

> **Investigate further:** The denominator is the *full* sample space (11), not just the converging subset — a non-converging k fails the condition rather than leaving the count. **Confirm that by listing which k converge and which of those clear the threshold. Then ask the trap question: how would the answer change if the problem said "given that the series converges"? Finally try the real TMUA version (2023 Paper 1 Q18)**: $a = 4$, $r = \frac{2k}{7}$, threshold $S > 3$, answer $\frac{5}{11}$.

4. The series $\sum_{k=1}^{\infty} ar^{k-1}$ converges (for real a, r) when $|r| < 1$. If a and r are instead allowed to be complex numbers, which is the correct convergence condition?

A) $|r| < 1$ (the same condition, using the complex magnitude).

B) $r < 1$ (a real inequality).

C) $a < 1$.

D) There is no meaningful convergence condition for complex r.

Answer: A) $|r| < 1$ (using the complex magnitude)

Method: The geometric series' convergence proof relies only on $|r^n| \to 0$, and $|r^n| = |r|^n$ holds identically whether r is real or complex — so the same magnitude condition transfers directly.

> **Investigate further:** Convergence is about the *size* of r shrinking powers to 0 — a notion indifferent to whether r is real or complex. **So describe the set of complex r for which the series converges, as a region of the plane. Then test $r = i$: what do the partial sums do, and is $|i| < 1$?**

5. Convert the recurring base-2 number $0.\overline{0111} = 0.0111\,0111\,0111\ldots$ (base 2) to a fraction in base 10. *(after TMUA 2023 Paper 2 Q15)*

A) $\frac{1}{3}$

B) $\frac{1}{5}$

C) $\frac{7}{15}$

D) $\frac{7}{16}$

Answer: C) $\dfrac{7}{15}$

Method: The repeating block 0111 (base 2) has integer value $0 \times 8 + 1 \times 4 + 1 \times 2 + 1 \times 1 = 7$, over a 4-bit block. This is a GP with $a = \dfrac{7}{16}$, $r = \dfrac{1}{16}$: $S_\infty = \dfrac{7/16}{1 - 1/16} = \dfrac{7/16}{15/16} = \dfrac{7}{15}$.

> **Investigate further:** Same mechanism as B8, with block value and place value both read in base 2. **State the general base-b rule: a block of length L with value v recurring gives which fraction?** Then check it reproduces both B8 (base 10) and this question, **and try the real TMUA version (2023 Paper 2 Q15), block** 0011**, answer** $\frac{1}{5}$.

6. Which of the following statements about arithmetic and geometric series is FALSE?

A) An AP's terms grow linearly in n; a GP's terms with $|r| > 1$ grow exponentially in n.

B) An AP diverges to $\pm\infty$ (or stays constant) as $n \to \infty$, unless $d = 0$.

C) A GP can converge to a finite sum to infinity even though no individual term is exactly zero, provided $|r| < 1$.

D) Every convergent GP has all its terms equal to zero eventually.

Answer: D) Every convergent GP has all its terms equal to zero eventually.

Method: A convergent GP's terms $ar^{n-1} \to 0$ as a *limit*, but for any finite n and $r \neq 0$, $ar^{n-1} \neq 0$ exactly — the terms get arbitrarily small without ever reaching zero. A), B), C) are all genuinely true statements.

> **Investigate further:** The same "limit vs. attained" idea as $0.\overline{9} = 1$: neither the terms' 0 nor the sum's S_∞ is reached at any finite step. **So identify the one degenerate case in which a GP's partial sums *do* literally attain S_∞ after finitely many terms.** There is exactly one such r, and it makes the series stop rather than converge.

7. Using $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$ from Day 2, evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{1}{5^k}$ by choosing an appropriate value of x.

A) $\frac{5}{4}$

B) $\frac{4}{5}$

C) 5

D) $\frac{1}{4}$

Answer: A) $\frac{5}{4}$

Method: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{5^k} = \sum_{k=0}^{\infty} \left(\frac{1}{5}\right)^k = (1-\tfrac{1}{5})^{-1}$ at $x = \frac{1}{5}$: $(1-\frac{1}{5})^{-1} = (\frac{4}{5})^{-1} = \dfrac{5}{4}$.

Investigate further: Recognising $\sum \frac{1}{5^k}$ as $\sum x^k$ at $x = \frac{1}{5}$ makes it a one-line substitution. **Now generalise: $\sum_{k=0}^{\infty} \frac{1}{b^k}$ for an integer $b \geq 2$ has a strikingly simple closed form — find it, and say what it tends to as b grows.** Test it at $b = 2$ against a sum you already know.

8. Let $S = 1+2x+3x^2+4x^3+\cdots$. Recall from Day 2 that this is exactly $(1-x)^{-2} = \sum_{k=0}^{\infty}(k+1)x^k$. Evaluate S at $x = \frac{1}{4}$.

A) $\frac{16}{9}$

B) $\frac{4}{3}$

C) $\frac{9}{16}$

D) $\frac{3}{4}$

Answer: A) $\frac{16}{9}$

Method: $(1-x)^{-2}$ at $x = \frac{1}{4}$: $(1-\frac{1}{4})^{-2} = (\frac{3}{4})^{-2} = \left(\frac{4}{3}\right)^2 = \dfrac{16}{9}$.

Investigate further: $1 + 2x + 3x^2 + \cdots$ is the term-by-term derivative of $1 + x + x^2 + \cdots$. **Push it one step further: differentiate again to find a closed form for $\sum_k (k+1)(k+2)x^k$, and check your result at $x = \frac{1}{4}$ against this question's answer.** Then predict the closed form of $(1-x)^{-m}$ without differentiating m times.

Section D Challenge

1. Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ exactly (not merely "approaches 1"), using the sum-to-infinity formula.

Then explain why the partial sum $S_n = \sum_{k=1}^{n} \dfrac{1}{2^k} = 1 - \dfrac{1}{2^n}$ never actually equals 1 for any finite n.

Answer: Proof via the sum-to-infinity formula; partial sums approach but never reach 1.

Method: $a = \frac{1}{2}, r = \frac{1}{2}$: $S_\infty = \dfrac{1/2}{1-1/2} = \dfrac{1/2}{1/2} = 1$ exactly, by the formula itself (not an approximation).

Separately, $S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{\frac{1}{2}(1-(\frac{1}{2})^n)}{\frac{1}{2}} = 1 - \dfrac{1}{2^n}$, which is strictly less than 1 for every finite n (since $\frac{1}{2^n} > 0$ always) — the infinite sum is the *limit* of these partial sums, not a partial sum itself.

Investigate further: The infinite sum is *defined* as the limit, and that limit is exactly 1 — no finite partial sum ever is. **Make the definition do visible work: given any tolerance $\varepsilon > 0$, find explicitly how many terms guarantee $1 - S_n < \varepsilon$.** Producing that n from ε is precisely what "the limit is 1" asserts.

2. Let $x = 0.\overline{9} = 0.999\ldots$ (a recurring 9). Writing x as an infinite geometric series and summing it, determine x exactly.

A) $x = 1$ exactly.

B) x is infinitesimally less than 1 but not equal to it.

C) $0.\overline{9}$ is undefined.

D) $x = \frac{9}{10}$.

Answer: A) $x = 1$ exactly.

Method: $0.\overline{9} = \dfrac{9}{10} + \dfrac{9}{100} + \cdots$, a GP with $a = \dfrac{9}{10}, r = \dfrac{1}{10}$. $S_\infty = \dfrac{9/10}{1 - 1/10} = \dfrac{9/10}{9/10} = 1$.

> **Investigate further:** There is no sense in which $0.\overline{9}$ is "slightly less than 1": it is the limit of its partial sums, and that limit is 1. **Test the objection directly — if $0.\overline{9} \neq 1$, their difference is a positive number, so name it.** Show no positive real can be smaller than every 10^{-n}, and the objection collapses.

3. A ball is dropped from height h. Each time it bounces, it rebounds to a fraction r of its previous height ($0 < r < 1$). Find, in terms of h and r, the total distance the ball travels (up and down, forever) before coming to rest.

A) $h \cdot \dfrac{1+r}{1-r}$

B) $h \cdot \dfrac{2}{1-r}$

C) $h \cdot \dfrac{1}{1-r}$

D) $h(1 + 2r)$

Answer: A) $h \cdot \dfrac{1+r}{1-r}$

Method: Total distance = (initial drop h) + (every rebound-and-fall pair, each pair covering $2\times$ that rebound's height). Rebound heights are $hr, hr^2, hr^3, \ldots$, a GP with first term hr, ratio r. Distance from rebounds $= 2 \times \dfrac{hr}{1-r}$. Total $= h + \dfrac{2hr}{1-r} = \dfrac{h(1-r) + 2hr}{1-r} = \dfrac{h(1+r)}{1-r}$.

> **Investigate further:** The factor of 2 per rebound is the easiest thing to forget — separate the unpaired initial drop from the infinitely many paired bounces. **Now sanity-check the formula at its two extremes: what total distance does it predict as $r \to 0$, and as $r \to 1^-$?** Both limits should match physical intuition; if one doesn't, the factor of 2 is in the wrong place.

4. The sum to infinity of a GP equals the sum of its first two terms. What can be said about the common ratio r?

A) $r = 0$ (the "GP" is actually constant).

B) $r = \pm 1$.

C) $r = \frac{1}{2}$.

D) No such r exists.

Answer: A) $r = 0$

Method: $S_\infty = a(1+r) \Rightarrow \dfrac{a}{1-r} = a(1+r) \Rightarrow \dfrac{1}{1-r} = 1+r$ (for $a \neq 0$) $\Rightarrow 1 = (1+r)(1-r) = 1 - r^2 \Rightarrow r^2 = 0 \Rightarrow r = 0$.

> **Investigate further:** $r = 0$ collapses the GP to its first term — degenerate, but the algebra forces it uniquely. **Try the natural follow-up: for which r does the sum to infinity equal the sum of the first *three* terms?** Set up the equation as before and see whether it again forces $r = 0$, or whether new solutions appear.

5. A sequence is defined by $a_1 = 3$ and $a_{n+1} = \frac{1}{2}a_n$ for $n \geq 1$. Using the sum-to-infinity formula, evaluate $\displaystyle\sum_{n=1}^{\infty} a_n$.

A) 6

B) 3

C) 4.5

D) It cannot be summed, since it's defined by a recurrence, not a formula.

Answer: A) 6

Method: The recurrence $a_{n+1} = \frac{1}{2}a_n$ with $a_1 = 3$ generates exactly the GP $3, \frac{3}{2}, \frac{3}{4}, \ldots$ (each term is half the last, matching $r = \frac{1}{2}$). $S_\infty = \dfrac{3}{1 - \frac{1}{2}} = 6$.

> **Investigate further:** "Multiply by a constant each time" is a GP in disguise — no new machinery needed. **So what happens if a constant is *added* as well, say $a_{n+1} = \frac{1}{2}a_n + 1$? Compute a few terms and see whether they still converge, and to what.** That single extra constant is exactly what Day 4 is built to handle.

Top 5 Patterns — Worksheet #3

1. **Sum to infinity: $S_\infty = \dfrac{a}{1-r}$, valid only for $|r| < 1$.** The central formula of the whole sheet. *(see A5, A7, A9, B3, B4)*

2. **Always check $|r| < 1$ before writing down S_∞ at all.** Convergence depends only on r, never on a. *(see A4, A8, B2, B7)*

3. **Recurring decimals are geometric series in disguise.** $a =$ the repeating block's value (as a fraction of its place value), $r = (\text{base})^{-(\text{block length})}$. *(see B8, C5, D2)*

4. **Day 2's $(1-x)^{-1} = \sum x^k$ *is* the geometric series.** Plug in any $|x| < 1$ to evaluate a numeric infinite series instantly. *(see B10, C7, C8)*

5. **Know which formula a question wants: finite $S_n = \dfrac{a(r^n - 1)}{r - 1}$ or infinite $S_\infty = \dfrac{a}{1-r}$.** They answer different questions and only one needs a convergence check. *(see A3, B1, B9, C2)*

Common Traps — Worksheet #3

1. **Applying $S_\infty = \frac{a}{1-r}$ without checking $|r| < 1$ first.** The formula still spits out a number even when the series actually diverges — that number is meaningless. *(see B7, C1)*

2. **Sign errors with negative common ratios inside $1 - r$.** $1 - (-\frac{1}{2}) = \frac{3}{2}$, not $\frac{1}{2}$ — expand

the double negative carefully. *(see A9, B9)*

3. **Confusing "converges" with "sums to something small."** A GP with r just under 1 still converges, but its sum can be enormous. *(see C6)*

4. **Treating a probability question's sample space as only the converging outcomes.** Non-convergent cases stay in the denominator — they just automatically fail the sum condition, they aren't removed from the count. *(see C3)*

5. **Assuming every recurrence relation needs unfamiliar machinery.** Some recurrences ("multiply by a constant each step") are secretly a GP, solvable with this sheet's tools already. *(see D5)*

"Every recurring decimal you have ever written down was a geometric series you declined to notice."

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths