Daily Sequences Drill #2 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Binomial & Trinomial Expansion · Pascal's Triangle as a Sequence · The Generalized Binomial Series.

Section A Rapid Recognition

1. Write down row 5 of Pascal's triangle (the coefficients of $(x+y)^5$).

Answer: $1, 5, 10, 10, 5, 1$

Method: Pascal's rule: each entry is the sum of the two above it. Row 4 is $1, 4, 6, 4, 1$; row 5 adds them pairwise with 1's at the ends.

> **Investigate further:** Row n has $n+1$ entries and is symmetric: $\binom{n}{k} = \binom{n}{n-k}$. **So when is the largest entry in a row unique, and when are there two equal largest entries?** Decide it from the parity of n, and say what $\binom{n}{k} = \binom{n}{n-k}$ forces at the exact middle.

2. Evaluate $\dbinom{6}{2}$.

Answer: 15

Method: $\binom{6}{2} = \frac{6\times5}{2!} = 15$.

> **Investigate further:** Equivalently $\binom{6}{2} = \binom{6}{4}$ by symmetry — always compute from the "short side". **Now find every pair (n, k) with $\binom{n}{k} = \binom{n}{k+1}$, i.e. two *adjacent* entries equal.** Set the two factorial expressions equal and the condition collapses to a single relation between n and k.

3. Write down $\displaystyle\sum_{k=0}^{n} \binom{n}{k}$ in closed form.

Answer: 2^n

Method: Setting $x = 1$ in $(1+x)^n = \sum_k \binom{n}{k} x^k$ gives $2^n = \sum_k \binom{n}{k}$.

> **Investigate further:** Substituting a number for x turns an identity in x into an identity in n — the most useful trick on the sheet. **You used $x = 1$ to get 2^n. What do $x = 2$ and $x = -2$ give, and what is the general closed form for $\sum_k \binom{n}{k} t^k$?** One substitution covers all three.

4. Expand $(1+x)^3$ fully.

Answer: $1 + 3x + 3x^2 + x^3$

Method: Row 3 of Pascal's triangle is $1, 3, 3, 1$.

> **Investigate further:** The coefficients are read straight off Pascal's triangle — no expanding by hand. **So, without expanding anything: what is the coefficient of x^3 in $(1+x)^3(1+x)^4$?** Combine the powers first. (The slow route — multiplying the two expansions and collecting terms — should give the same number; it is worth doing once to see how much work the shortcut saves.)

5. Find the coefficient of x^2 in $(1+x)^6$.

Answer: 15

Method: $\binom{6}{2} = 15$ (row 6: $1, 6, 15, 20, 15, 6, 1$).

> **Investigate further:** Same computation as A2 — spotting that saves real time. **A deeper question: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ involves dividing by $k!(n-k)!$, yet the result is always a whole number. Why must that division always come out exact?** A counting argument answers it instantly; pure algebra struggles.

6. Row 4 of Pascal's triangle is $1, 4, 6, 4, 1$. Is this sequence of five numbers arithmetic? Justify with one check.

Answer: No.

Method: Consecutive differences are $4-1=3$, $6-4=2$, $4-6=-2$, $1-4=-3$ — not constant, so the sequence is not arithmetic.

> **Investigate further:** Rows are symmetric and unimodal (rise then fall), which rules out any row of more than 2 entries being arithmetic. **Can a row ever be *geometric*?** Write the ratio $\binom{n}{k+1}/\binom{n}{k}$ as a formula in n and k, then ask what forcing it to be constant in k would require.

7. Simplify $\displaystyle\binom{n}{0} + \binom{n}{1}$.

Answer: $1+n$

Method: $\binom{n}{0} = 1$, $\binom{n}{1} = n$, so the sum is $1+n$.

> **Investigate further:** This is the first two terms of the row summing to 2^n. **For which n do those first two entries alone account for more than half the row — that is, $1+n > 2^{n-1}$?** Only a couple of small n survive; check them directly rather than solving the inequality.

8. State the value of $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}$ for $n \geq 1$.

Answer: 0

Method: Setting $x=-1$ in $(1+x)^n$: $(1-1)^n = 0^n = 0$ for $n \geq 1$, and this equals $\sum_k (-1)^k \binom{n}{k}$.

> **Investigate further:** The twin of A3: $x=1$ gives the row sum, $x=-1$ gives the alternating sum 0. **But the alternating sum is 0 only for $n \geq 1$ — what does it come to at $n=0$, and why does the argument break there?** Look at what $(1-1)^n$ actually evaluates to when $n=0$.

9. Find the constant term in $(1+x)^5$.

Answer: 1

Method: The constant term of $(1+x)^5$ is the x^0 term, $\binom{5}{0} = 1$.

> **Investigate further:** Every $(1+x)^n$ has constant term 1, since $\binom{n}{0}=1$. **Now consider $(1+x)^n(1-x)^n$. What is its constant term, and what is its coefficient of x^2?** Combine into a single power of $(1-x^2)$ first — the answer is much cleaner than multiplying two expansions.

10. True or false: row n of Pascal's triangle always has exactly $n+1$ entries.

Answer: True

Method: Row n has entries $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ — that's $n+1$ entries, from $k=0$ to $k=n$ inclusive.

> **Investigate further:** A common slip is saying row n has n entries — count inclusively from 0. **So how many entries are there in rows 0 through n altogether?** You should recognise the answer as a familiar sequence from Day 1 — and it is worth noticing *why* a triangle of numbers totals a triangular number.

Section B Manipulation Drills

1. Prove that $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n$ by substituting a value into the expansion of $(1+x)^n$.

Answer: Proof by substitution $x=1$.

Method: $(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$. Setting $x=1$: $(1+1)^n=2^n=\sum_{k=0}^{n}\binom{n}{k}(1)^k=\sum_{k=0}^{n}\binom{n}{k}$. $\square$

> **Investigate further:** The cleanest proof of the identity: one substitution, no induction. **Try to prove the same identity a second way, by induction on n using Pascal's rule $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$.** Comparing the two proofs shows exactly what the substitution bought you.

2. Row 10 of Pascal's triangle sums to 2^{10}. If instead you sum only its first four entries, $\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\binom{10}{3}$, is the result arithmetic, geometric, or neither, as a rule? Compute the value.

Answer: Neither in general; the value is 176.

Method: $\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\binom{10}{3}=1+10+45+120=176$. The four terms themselves $(1,10,45,120)$ have differences $9,35,75$ (not constant) and ratios $10,4.5,2.\overline{6}$ (not constant), so the partial sum follows no simple pattern.

> **Investigate further:** Individual entries in a row are neither arithmetic nor geometric; only the full row sum (2^n) and alternating sum (0) are clean. **What about the sum of just the even-indexed entries, $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots$?** Add the $x=1$ and $x=-1$ substitutions together and see what survives.

3. Using $x=-1$ in $(1+x)^n$ $(n\geq 1)$, what is $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}$?

A) 0

B) 1

C) 2^n

D) $(-1)^n$

Answer: A) 0

Method: $(1+(-1))^n = 0^n = 0$ for $n \geq 1$.

> **Investigate further:** At $n=0$ the sum is $\binom{0}{0} = 1$, not 0 — the identity needs $n \geq 1$. **Make a habit of it: for each identity on this sheet, test $n=0$ and $n=1$ before trusting it.** Which of today's identities actually survive $n=0$ intact, and which need the restriction stated?

4. Find the coefficient of x^3 in $(2+x)^5$.

A) 10

B) 40

C) 80

D) 20

Answer: B) 40

Method: General term: $\binom{5}{k}2^{5-k}x^k$. At $k=3$: $\binom{5}{3}2^2 = 10 \times 4 = 40$.

> **Investigate further:** Don't drop the 2^{5-k} factor — answering $\binom{5}{3} = 10$ treats this as $(1+x)^5$. **Which term of $(2+x)^5$ is the largest when $x=1$, and does the largest coefficient sit at the same place as the largest *term*?** They are not the same question, and that is the point.

5. Find the coefficient of x^4 in $(1-2x)^6$.

A) 60

B) -240

C) 240

D) -60

Answer: C) 240

Method: General term: $\binom{6}{k}(-2x)^k$. At $k=4$: $\binom{6}{4}(-2)^4 = 15 \times 16 = 240$. The sign is positive since $(-2)^4$ is positive (even power).

> **Investigate further:** The trap is -240, from forgetting an even power of a negative is positive. **State the general sign rule: in $(a-b)^n$, exactly which terms are negative?** Give the condition in terms of the index k, then check it against this expansion.

6. Find the constant term in the expansion of $\left(x + \dfrac{1}{x}\right)^4$.

A) 4

B) 6

C) 8

D) 1

Answer: B) 6

Method: General term: $\binom{4}{k}x^{4-k}x^{-k} = \binom{4}{k}x^{4-2k}$. Constant term needs $4-2k=0 \Rightarrow k=2$: $\binom{4}{2} = 6$.

> **Investigate further:** With both positive and negative powers of x, set the total exponent to 0 and solve for k first. **Now ask when that method returns nothing: for which powers n does $\left(x+\frac{1}{x}\right)^n$ have no constant term at all?** Work out the exponent equation in general and see what integrality of k demands.

7. In the expansion of $(1+x)^n$, the coefficient of x^2 is 21 times the coefficient of x^1. Find n.

A) 21

B) 42

C) 43

D) 44

Answer: C) 43

Method: $\dfrac{\binom{n}{2}}{\binom{n}{1}} = \dfrac{n-1}{2} = 21 \Rightarrow n-1 = 42 \Rightarrow n = 43.$

> **Investigate further:** The ratio trick $\binom{n}{2}/\binom{n}{1} = \frac{n-1}{2}$ beats writing both coefficients out. **Find the general ratio $\binom{n}{k+1}/\binom{n}{k}$ as a formula in n and k, then use it to say exactly where a row stops increasing and starts decreasing.** That single formula locates the peak of any row.

8. The coefficients of x^2 and x^3 in the expansion of $(1+x)^n$ are equal. Find n.

A) 4

B) 5

C) 6

D) Cannot be determined.

Answer: B) 5

Method: $\binom{n}{2} = \binom{n}{3} \Rightarrow$ (by the symmetry $\binom{n}{k} = \binom{n}{n-k}$) 2 and 3 must be complementary: $n-2=3 \Rightarrow n=5$. Check: $\binom{5}{2} = 10 = \binom{5}{3}$. ✓

> **Investigate further:** This works because $2 \neq 3$ forces the symmetry route; equal exponents would make any n work and the question ill-posed. **So state the general rule: given $\binom{n}{a} = \binom{n}{b}$ with $a \neq b$, what is n?** One equation, and it should need no case analysis.

9. Find the coefficient of x^2 in the expansion of $(1+2x+3x^2)^3$.

A) 15

B) 18

C) 21

D) 24

Answer: C) 21

Method: Trinomial theorem: terms $(1)^a(2x)^b(3x^2)^c$ with $a+b+c=3$, exponent $b+2c=2$. Case $(b,c)=(2,0), a=1$: coefficient $\frac{3!}{1!2!0!}2^2 = 3\times 4 = 12$. Case $(b,c)=(0,1), a=2$: coefficient $\frac{3!}{2!0!1!}3^1 = 3\times 3 = 9$. Total: $12+9=21$.

> **Investigate further:** Every (a, b, c) with $a + b + c = 3$ and $b + 2c = 2$ must be found systematically — missed cases are the classic trinomial error. **Impose a discipline: solve for one variable, list its possible values in order, and stop only when the remaining variables go negative. How many triples are there here, and how do you *know* the list is complete?**

10. The row-sums of Pascal's triangle are $1, 2, 4, 8, 16, \ldots$ Is this sequence of row-sums arithmetic, geometric, both, or neither? State the relevant ratio or difference.

Answer: Geometric with common ratio 2.

Method: Row sums are $2^0, 2^1, 2^2, \ldots = 1, 2, 4, 8, 16, \ldots$ — each term is exactly double the previous one.

> **Investigate further:** In sequence language: the row sums form a GP even though no single row's entries do. **What is the sum of *all* entries in rows 0 through n, and is that total itself geometric, arithmetic, or neither?** Sum the row sums — Day 3 will hand you the formula, but you can already guess its shape.

Section C Substitution & Structure

1. Which of the following statements about partial sums of binomial coefficients is correct?

A) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k-1}$ always equals 2^n, for any $k \leq n$.

B) There is no simple closed form for a general partial sum of binomial coefficients, unlike the full row sum.

C) Partial sums of binomial coefficients are always arithmetic in k.

D) Partial sums of binomial coefficients are always geometric in k.

Answer: B) There is no simple closed form for a general partial sum of binomial coefficients, unlike the full row sum.

Method: The full sum ($k = n + 1$ terms) collapses via $x = 1$ substitution to 2^n; a partial sum has no equivalent single substitution trick, and in general must be computed term by term (as in B2's 176).

> **Investigate further:** The exception worth knowing: summing down a *column* across rows does have a closed form — the hockey-stick identity (C3). **Before reading C3, guess it: compute $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2}$ and see which single binomial coefficient it equals.** The pattern is visible from one example.

2. Find the coefficient of x in the expression $(1 + x)^0 + (1 + x)^1 + \cdots + (1 + x)^{40}$. *(after TMUA 2019 Paper 1 Q3)*

A) 40

B) 41

C) 820

D) 1640

E) 1681

Answer: C) 820

Method: The coefficient of x in $(1 + x)^k$ is $\binom{k}{1} = k$. Summing over $k = 0$ to 40: $\sum_{k=0}^{40} k = \frac{40 \times 41}{2} = 820$.

> **Investigate further:** Day 1's AP sum formula, now extracting information from a sum of binomial expansions — the two toolkits meet here. **Try the real TMUA version (2019 Paper 1 Q3), which sums to $k = 80$ instead of 40.** Predict the answer from the structure before computing it, then check you get $\frac{80\times 81}{2} = 3240$.

3. Find, in terms of n, the coefficient of x^2 in the expression $(1+x)^0 + (1+x)^1 + \cdots + (1+x)^n$. *(after TMUA 2019 Paper 1 Q3)*

A) $\binom{n}{3}$

B) $\binom{n+1}{3}$

C) $\binom{n+1}{2}$

D) $2^n - n - 1$

Answer: B) $\binom{n+1}{3}$

Method: The coefficient of x^2 in $(1+x)^k$ is $\binom{k}{2}$ (zero for $k < 2$). Summing $\sum_{k=2}^{n} \binom{k}{2} = \binom{n+1}{3}$ is the hockey-stick identity: summing a column of Pascal's triangle from its start lands one row down and one column right.

> **Investigate further: Prove the hockey-stick identity by induction on n using Pascal's rule, then explain why C2 is the same identity one column to the left** (with $\binom{k}{1} = k$ in place of $\binom{k}{2}$). **Finally: what does the column one step further right, $\binom{k}{3}$, sum to?** Extend the pattern and verify it on a small case.

4. In the expansion of $(1 + x + x^2)^4$, what is the coefficient of x^3?

A) 12

B) 16

C) 20

D) 24

Answer: B) 16

Method: Trinomial theorem on $(1 + x + x^2)^4$: terms $(1)^a(x)^b(x^2)^c$, $a + b + c = 4$, exponent $b + 2c = 3$. Case $(b, c) = (3, 0), a = 1$: $\frac{4!}{1!3!0!} = 4$. Case $(b, c) = (1, 1), a = 2$: $\frac{4!}{2!1!1!} = 12$. Total: $4 + 12 = 16$.

> **Investigate further:** Same systematic case-finding as B9 at a larger power — list every (b, c) pair before computing any coefficient. **How many terms does the full expansion of $(1 + x + x^2)^n$ have before collecting like terms, and how many distinct powers of x can appear after collecting?** The second count is far smaller, and knowing it in advance tells you how much casework to expect.

5. Which of the following is the expansion of $(1 - x)^{-1}$ as an infinite series in x?

A) $1 - x + x^2 - x^3 + \cdots$

B) $1 + x + x^2 + x^3 + \cdots$

C) $1 - 2x + 3x^2 - 4x^3 + \cdots$

D) It cannot be expanded as a series.

Answer: B) $1 + x + x^2 + x^3 + \cdots$

Method: Long division, or the generalized binomial series $(1-x)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} x^k$ with $(-x)^k$ sign conventions, both give every coefficient equal to 1.

> **Investigate further:** This is the geometric series with $a = 1, r = x$ in disguise, and the whole of Day 3 is built on it. **Verify it by multiplying out** $(1-x)(1+x+x^2+\cdots)$ **and watching it telescope to** 1 **— then say precisely which values of** x **make that manipulation legitimate.** The algebra looks valid for every x; it is not.

6. The generalized binomial expansion gives $(1-x)^{-2} = \sum_{k=0}^{\infty}(k+1)x^k$. Find the coefficient of x^3.

A) 3

B) 4

C) 6

D) 10

Answer: B) 4

Method: At $k = 3$: coefficient is $k + 1 = 4$.

> **Investigate further:** The coefficients $k+1$ form an AP with $d = 1$: differentiating $(1-x)^{-1}$ turned a constant coefficient sequence into a linear one. **So predict, before computing: what sort of sequence are the coefficients of** $(1-x)^{-3}$**, and what is the coefficient of** x^k**?** Differentiate once more and check your prediction against C8.

7. Which of the following is FALSE, for a positive integer n?

A) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$

B) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

C) $\displaystyle\binom{n}{k} = \binom{n}{n-k}$ for all $0 \le k \le n$

D) $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = 2^n$

Answer: D) $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = 2^n$

Method: The true identity is $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$ (proved in D3), not 2^n. Check $n = 3$: true value is $3 \times 4 = 12$, but D claims $2^3 = 8 \neq 12$.

> **Investigate further:** Three genuine identities beside one plausible fabrication — the defence is always to test a small case. **Take the false option and find the smallest** n **at which it fails.** Then make it a habit: for any half-remembered binomial identity, which single value of n is fastest to

> test, and which is most likely to be misleading?

8. The infinite series $1 + x + x^2 + x^3 + \cdots$ equals $(1-x)^{-1}$. Using $(1-x)^{-3} = \sum_{k=0}^{\infty} \binom{k+2}{2} x^k$, find the coefficient of x^5 in $(1-x)^{-3}$.

A) $\binom{5}{2}$

B) $\binom{6}{2}$

C) $\binom{7}{2}$

D) $\binom{8}{2}$

Answer: C) $\binom{7}{2}$

Method: $(1-x)^{-3} = \sum_{k=0}^{\infty} \binom{k+2}{2} x^k$. At $k = 5$: $\binom{5+2}{2} = \binom{7}{2} = 21$.

> **Investigate further:** Across $(1-x)^{-1}, (1-x)^{-2}, (1-x)^{-3}$ the coefficient of x^k is $\binom{k}{0}, \binom{k+1}{1}, \binom{k+2}{2}$ — "stars and bars in series form". **Write down the coefficient of x^k in $(1-x)^{-m}$ for general m, then check your formula reproduces all three cases above.** Test $m = 1$ especially — it is the one most likely to expose an off-by-one.

Section D Challenge

1. Find the coefficient of x^2y^3 in the expansion of $(1 + x + y^2)^5$. *(after TMUA 2020 Paper 1 Q13)*

A) 0

B) 10

C) 20

D) 30

E) 60

Answer: A) 0

Method: General term of $(1 + x + y^2)^5$: $(1)^a(x)^b(y^2)^c$ with $a + b + c = 5$, contributing $x^b y^{2c}$. The power of y is always $2c$ — an even number. Since 3 is odd, there is no valid (a, b, c) giving x^2y^3, so its coefficient is 0.

> **Investigate further:** The reward here is recognising an *impossible* target instantly instead of grinding through cases to find them all empty. **Make the parity argument general: in $(1 + x + y^2)^n$, exactly which coefficients of $x^a y^b$ are forced to be zero?** State the rule in one line, then use it to answer the real TMUA version (2020 Paper 1 Q13), which asks for the achievable x^2y^4.

2. The coefficient of x^2 in $(1 + x + 2x^2)^3$ equals the coefficient of x^2 in $(1 - ax)^3$. Find the possible value(s) of a. *(after TMUA Practice Paper 1 Q19)*

A) $a = \pm\sqrt{3}$

B) $a = \pm 3$

C) $a = \sqrt{3}$ only

D) $a = 3$ only

Answer: A) $a = \pm\sqrt{3}$

Method: Coefficient of x^2 in $(1+x+2x^2)^3$: cases $(b,c) = (2,0), a{=}1$ gives $\frac{3!}{1!2!0!}1^2 = 3$; $(b,c) = (0,1), a{=}2$ gives $\frac{3!}{2!0!1!}2^1 = 6$. Total $3+6=9$. Coefficient of x^2 in $(1-ax)^3$: $\binom{3}{2}(-a)^2 = 3a^2$. Setting equal: $3a^2 = 9 \Rightarrow a^2 = 3 \Rightarrow a = \pm\sqrt{3}$.

> **Investigate further: Work through the real version by hand (Practice Paper 1 Q19): match the coefficient of x^3 in $(1+2x+3x^2)^6$ against *twice* the coefficient of x^4 in $(1-ax^2)^5$, and confirm $a = \pm\sqrt{17}$.** Then explain why a comes out as $\pm$ a surd here while the sheet's version gives $\pm\sqrt{3}$ — what feature of the equation guarantees two solutions rather than one?

3. Prove that $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$ for every positive integer n, by differentiating the binomial expansion of $(1+x)^n$.

Answer: Proof by differentiating $(1+x)^n$.

Method: $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$. Differentiating both sides with respect to x: $n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}$ (the $k=0$ term vanishes since it has no x). Setting $x=1$: $n \cdot 2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}$. $\square$

> **Investigate further:** The standard move for weighted binomial sums: differentiate first, substitute second. **Now evaluate $\sum_{k=0}^{n} k^2\binom{n}{k}$ the same way** — differentiate $x \cdot n(1+x)^{n-1}$ once more before substituting. **Check your answer at $n=2$ by hand**, where the sum is short enough to verify term by term.

4. Expanding $(1+x)^n(1+x)^m$ two different ways — first by combining the factors as $(1+x)^{n+m}$, then by multiplying the two separate expansions together and comparing the coefficient of x^k on both sides — produces which identity?

A) Vandermonde's identity: $\displaystyle\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$.

B) $\dbinom{n}{k} + \dbinom{m}{k} = \dbinom{n+m}{k}$.

C) $\dbinom{n}{k}\dbinom{m}{k} = \dbinom{n+m}{k}$.

D) No identity results — the two sides are unrelated.

Answer: A) Vandermonde's identity.

Method: $(1+x)^n(1+x)^m = \left(\sum_i \binom{n}{i}x^i\right)\left(\sum_j \binom{m}{j}x^j\right)$. The coefficient of x^k on the right is $\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}$ (pairing every way $i+j=k$). Since the left side is also $(1+x)^{n+m}$, whose coefficient of x^k is $\binom{n+m}{k}$, the two must be equal.

> **Investigate further:** Vandermonde also reads combinatorially: choose k people from n men and m women, split by how many men. **Set $m=n$ and $k=n$ to get a striking special case — what single binomial coefficient does $\sum_k \binom{n}{k}^2$ equal?** Use the symmetry $\binom{n}{k} = \binom{n}{n-k}$ to see why the squares appear, then verify at $n=3$.

5. The coefficients of $(1-x)^{-1}$ are all 1 (constant in k). The coefficients of $(1-x)^{-2}$, namely $(k+1)$, form an arithmetic sequence in k. What kind of sequence, as a function of k, do the coefficients of $(1-x)^{-3} = \sum_{k=0}^{\infty} \binom{k+2}{2} x^k$ form?

A) Arithmetic.

B) Geometric.

C) Quadratic in k (neither arithmetic nor geometric).

D) Constant.

Answer: C) Quadratic in k (neither arithmetic nor geometric).

Method: $\binom{k+2}{2} = \dfrac{(k+2)(k+1)}{2} = \dfrac{1}{2}k^2 + \dfrac{3}{2}k + 1$, a genuine quadratic in k (nonzero k^2 coefficient).

> **Investigate further:** Day 1's "S_n is quadratic" pattern in disguise: the coefficients of $(1-x)^{-1}, (1-x)^{-2}, (1-x)^{-3}$ are degree $0, 1, 2$ in k, each inverse power raising the degree by one. **Is that a coincidence, or the same fact twice?** Consider what multiplying a series by $(1-x)^{-1}$ does to its coefficients — and compare that operation with taking partial sums of a sequence.

Top 5 Patterns — Worksheet #2

1. **Row sum: $\sum_k \binom{n}{k} = 2^n$, from setting $x = 1$ in $(1+x)^n$.** The single most useful substitution in the whole sheet. *(see A3, B1, C2)*

2. **Alternating sum: $\sum_k (-1)^k \binom{n}{k} = 0$ for $n \geq 1$, from setting $x = -1$.** The "twin" of the row-sum trick. *(see A8, B3)*

3. **The generalized binomial series bridges straight into geometric series.** $(1-x)^{-m} = \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} x^k$ — at $m = 1$ this is exactly the geometric series. *(see C5, C6, C8, D5)*

4. **Trinomial/multinomial coefficient extraction: find every (a, b, c) with $a+b+c =$ power and the right exponent.** Missing a case is the single most common error. *(see B9, C4, D1, D2)*

5. **Differentiate $(1+x)^n$ term-by-term to extract weighted sums like $\sum k\binom{n}{k}$.** Substitute $x = 1$ after differentiating, not before. *(see C7, D3)*

Common Traps — Worksheet #2

1. **Confusing a partial binomial sum with the full row sum.** Only the complete sum $k = 0$ to n has the clean 2^n closed form — a partial sum in general has no shortcut. *(see C1)*

2. **Assuming every target power is achievable in a trinomial expansion.** $(1 + x + y^2)^n$ can only ever produce *even* powers of y — an odd target power isn't a computation to grind through, it's genuinely zero. *(see D1)*

3. **Off-by-one errors between $\binom{n+1}{3}$ and $\binom{n}{3}$ in hockey-stick-style partial sums.** Summing a column of Pascal's triangle lands one row *down*, not on the same row. *(see C3)*

4. **Assuming binomial-series coefficients stay linear (arithmetic) in k for every power of $(1-x)^{-m}$.** Only $m = 2$ gives an AP — higher m gives higher-degree polynomials in k. *(see D5)*

5. **Forgetting the multinomial coefficient needs all three factorials, $\frac{n!}{p!q!r!}$, not just a**

single $\binom{n}{k}$. A trinomial expansion is not two nested binomial expansions in disguise. *(see B9, C4)*

"Two substitutions, $x = 1$ and $x = -1$, did most of today's work. The rest was you expanding brackets by hand."

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths