Daily Number Theory Drill #7 — Answers & Investigations

Sheet by David Akinyele-Aje

Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Combinatorial Number Theory · Divisor Counting · Synoptic Mastery.

Section A Rapid Recognition

($\leq$15 s each)

Primes, moduli, and basics. Pure recognition.

1. Find the prime factorisation of 91.

Answer: 7×13

Method: Testing small primes: 91 is not divisible by 2, 3, or 5. Testing 7 gives $91 \div 7 = 13$, which is also prime.

> **Investigate further:** Can you quickly spot why 1001 is a multiple of 91? Hint: $1001 = 7 \times 11 \times 13$.

2. What is the last digit of 3^{20}?

Answer: 1

Method: The powers of 3 follow a cycle modulo 10: $3^1 = 3$, $3^2 = 9$, $3^3 = 27 \equiv 7$, $3^4 = 81 \equiv 1$. Since 20 is a multiple of 4, $3^{20} \equiv 1 \pmod{10}$.

> **Investigate further:** What is the last digit of 3^{21}?

3. Find $\gcd(144, 60)$.

Answer: 12

Method: Using prime factorisation: $144 = 2^4 \times 3^2$ and $60 = 2^2 \times 3 \times 5$. The greatest common divisor is $2^2 \times 3 = 12$.

> **Investigate further:** Can you find the lowest common multiple of 144 and 60 using the GCD?

4. Compute $12 \times 13 \pmod 5$.

Answer: 1

Method: Reduce each factor modulo 5 first: $12 \equiv 2 \pmod 5$ and $13 \equiv 3 \pmod 5$. Then $2 \times 3 = 6 \equiv 1 \pmod 5$.

> **Investigate further:** Can you compute $12^{13} \pmod 5$?

5. What is the smallest prime factor of 323?

Answer: 17

Method: Note that 323 is very close to $324 = 18^2$. We can write $323 = 18^2 - 1 = (18-1)(18+1) = 17 \times 19$. Both 17 and 19 are prime, so the smallest is 17.

> **Investigate further:** Can you use difference of two squares to factorise 899?

6. How many positive divisors does 28 have?

Answer: 6

Method: The prime factorisation is $28 = 2^2 \times 7^1$. The number of divisors is found by adding 1 to each exponent and multiplying: $(2+1)(1+1) = 3 \times 2 = 6$.

> **Investigate further:** Can you find the sum of these 6 divisors?

7. Compute $5^3 \pmod 6$.

Answer: 5

Method: Note that $5 \equiv -1 \pmod 6$. Therefore, $5^3 \equiv (-1)^3 = -1 \equiv 5 \pmod 6$.

> **Investigate further:** What is $5^{100} \pmod 6$?

8. Find $x \pmod 7$ if $4x \equiv 1 \pmod 7$.

Answer: 2

Method: We want a multiple of 4 that is 1 more than a multiple of 7. Testing multiples of 4: $4, 8, 12, \ldots$. We see that $8 \equiv 1 \pmod 7$, so $4x \equiv 8 \pmod 7$, giving $x \equiv 2 \pmod 7$.

> **Investigate further:** Can you solve $4x \equiv 3 \pmod 7$?

9. What is the remainder when 10^{10} is divided by 9?

Answer: 1

Method: Since $10 \equiv 1 \pmod 9$, raising both sides to the power of 10 gives $10^{10} \equiv 1^{10} = 1 \pmod 9$.

> **Investigate further:** What is the remainder when 8^{10} is divided by 9?

10. Which is larger: 2^{30} or 3^{20}?

Answer: 3^{20}

Method: Rewrite the expressions with a common exponent: $2^{30} = (2^3)^{10} = 8^{10}$ and $3^{20} = (3^2)^{10} = 9^{10}$. Since $9 > 8$, it follows that $9^{10} > 8^{10}$.

> **Investigate further:** Which is larger: 2^{50} or 5^{20}?

Section B Manipulation Drills ($\leq$50 s each)

Index laws, modular equations, and divisor counting.

1. Find the last two digits of 7^4.

Answer: 01

Method: We compute $7^2 = 49$, and then $7^4 = 49^2$. Since $49 = 50 - 1$, $49^2 = 2500 - 100 + 1 = 2401$. The last two digits are 01.

> **Investigate further:** What are the last two digits of 7^{400}?

2. How many trailing zeros are there in 20!?

Answer: 4

Method: Trailing zeros are produced by factors of 10, which require pairs of 2s and 5s. Since 2s are more abundant, we just count the number of factors of 5 in 20!. This is $\lfloor 20/5 \rfloor = 4$.

> **Investigate further:** How many trailing zeros are in 100!?

3. Find the smallest positive integer n such that $120n$ is a perfect square.

Answer: 30

Method: The prime factorisation of 120 is $2^3 \times 3^1 \times 5^1$. For $120n$ to be a perfect square, all prime exponents must be even. Thus, we need to multiply by $2^1 \times 3^1 \times 5^1 = 30$.

> **Investigate further:** What is the smallest n such that $120n$ is a perfect cube?

4. Find the largest power of 3 that divides 50!.

Answer: 22

Method: By Legendre's formula, the exponent of 3 in 50! is $\lfloor 50/3 \rfloor + \lfloor 50/9 \rfloor + \lfloor 50/27 \rfloor = 16 + 5 + 1 = 22$.

> **Investigate further:** What is the largest power of 9 that divides 50!?

5. Find the unique pair of positive integers (x, y) such that $x^2 - y^2 = 17$.

Answer: $(9, 8)$

Method: Factorising gives $(x - y)(x + y) = 17$. Since 17 is prime and x, y are positive integers, $x + y > x - y$, so we must have $x - y = 1$ and $x + y = 17$. Adding the equations gives $2x = 18 \implies x = 9$, and $y = 8$.

> **Investigate further:** Can you find a pair of positive integers such that $x^2 - y^2 = 18$?

6. What is the remainder when 3^{100} is divided by 10?

Answer: 1

Method: The question asks for the last digit of 3^{100}. We know $3^4 = 81 \equiv 1 \pmod{10}$. Therefore, $3^{100} = (3^4)^{25} \equiv 1^{25} = 1 \pmod{10}$.

> **Investigate further:** What is the remainder when 3^{100} is divided by 7?

7. Solve for x in the range $0 \leq x < 11$: $3x + 4 \equiv 6 \pmod{11}$.

Answer: 8

Method: Subtract 4 from both sides to get $3x \equiv 2 \pmod{11}$. We can add multiples of 11 to 2 until we get a multiple of 3: $2 + 11 = 13$, $13 + 11 = 24$. Thus $3x \equiv 24 \pmod{11}$, which gives $x \equiv 8 \pmod{11}$.

> **Investigate further:** Can you solve $5x \equiv 7 \pmod{11}$?

8. Find the sum of the positive divisors of 50.

Answer: 93

Method: The prime factorisation is $50 = 2^1 \times 5^2$. The sum of divisors is given by $(1+2^1)(1+5^1+5^2) = 3 \times 31 = 93$.

Investigate further: Find the sum of the positive divisors of 100.

9. How many prime numbers are there strictly between 40 and 50?

Answer: 3

Method: The odd numbers between 40 and 50 are 41, 43, 45, 47, 49. 45 is a multiple of 5, and 49 is 7^2. The remaining numbers, 41, 43, and 47, are not divisible by 2, 3, 5, or 7, so they are prime.

Investigate further: How many prime numbers are there between 90 and 100?

10. Find the smallest three-digit positive integer that leaves a remainder of 2 when divided by $3, 4$, and 5.

Answer: 122

Method: Let the number be x. Then $x-2$ is divisible by 3, 4, and 5. The smallest positive common multiple is $\text{lcm}(3,4,5) = 60$. Thus $x - 2 = 60k$, so $x = 60k + 2$. Since we need the smallest three-digit integer, we set $k = 2$, which gives $x = 120 + 2 = 122$.

Investigate further: What is the smallest four-digit positive integer that satisfies this property?

Section C Substitution & Structure (≤90 s each)

Synoptic links connecting number theory to algebra and combinatorics.

1. Find the product of all positive integers $n < 50$ that have exactly three positive divisors. *(after SMC 2015 Q16)*

Answer: 44100

Method: A number has exactly three divisors if and only if it is the square of a prime number ($n = p^2$), so its divisors are $1, p, p^2$. The primes p for which $p^2 < 50$ are $2, 3, 5, 7$. Their squares are $4, 9, 25, 49$. The product is $4 \times 9 \times 25 \times 49 = 100 \times 441 = 44100$.

Investigate further: This question builds upon the archetype of mapping divisor counts to prime factorisations, as seen in SMC 2015 Q16. Can you find the sum of all numbers less than 100 with exactly three divisors?

2. Find the smallest positive integer n such that n has exactly 15 positive divisors. *(after SMC 2018 Q20)*

Answer: 144

Method: The number of divisors is given by the product of one more than each prime exponent. Since $15 = 15$ or 3×5, n must be of the form p^{14} or p^4q^2 for distinct primes p, q. For p^{14}, the smallest is $2^{14} = 16384$. For p^4q^2, we minimise by assigning the larger exponent to the smaller prime: $2^4 \times 3^2 = 16 \times 9 = 144$. The smallest value is 144.

Investigate further: This problem tests optimisation within divisor counting, contrasting with direct prime evaluation in SMC 2018 Q20. What is the smallest positive integer with exactly 14 positive divisors?

3. How many positive divisors of $3^5 \times 5^4 \times 7^3$ are multiples of 45? *(after SMC 2021 Q19)*

Answer: 64

Method: Any divisor of $3^5 \times 5^4 \times 7^3$ must be of the form $3^a 5^b 7^c$ where $0 \leq a \leq 5$, $0 \leq b \leq 4$, and $0 \leq c \leq 3$. For the divisor to be a multiple of $45 = 3^2 \times 5^1$, we require $a \geq 2$ and $b \geq 1$. Thus, the choices are $a \in \{2, 3, 4, 5\}$ (4 options), $b \in \{1, 2, 3, 4\}$ (4 options), and $c \in \{0, 1, 2, 3\}$ (4 options). The total number of such divisors is $4 \times 4 \times 4 = 64$.

> **Investigate further:** This explores combinatorics over prime exponents, an extension of direct combinations tested in SMC 2021 Q19. How many divisors of the same number are perfect squares?

4. How many positive divisors of 10! are perfect squares?

Answer: 30

Method: The prime factorisation of 10! is $2^8 \times 3^4 \times 5^2 \times 7^1$. A divisor is a perfect square if all its prime exponents are even. The exponent of 2 can be $0, 2, 4, 6, 8$ (5 choices). The exponent of 3 can be $0, 2, 4$ (3 choices). The exponent of 5 can be $0, 2$ (2 choices). The exponent of 7 must be 0 (1 choice). The total number of square divisors is $5 \times 3 \times 2 \times 1 = 30$.

> **Investigate further:** How many positive divisors of 10! are perfect cubes?

5. What is the greatest possible value of $\gcd(n^2 + 5, n + 2)$ for a positive integer n?

Answer: 9

Method: Use the Euclidean algorithm: $\gcd(n^2 + 5, n + 2) = \gcd(n^2 + 5 - (n - 2)(n + 2), n + 2) = \gcd(n^2 + 5 - (n^2 - 4), n + 2) = \gcd(9, n + 2)$. The greatest possible value is 9, which occurs when $n + 2$ is a multiple of 9, such as $n = 7$.

> **Investigate further:** What is the greatest possible value of $\gcd(n^3 + 1, n + 2)$?

6. Find the sum of all digits of the number $10^{15} - 2026$.

Answer: 126

Method: The number 10^{15} is a 1 followed by 15 zeros. Subtracting 2026 gives 999999999997974, which has 15 digits in total. The first 11 digits are 9s. The sum of the digits is $11 \times 9 + 7 + 9 + 7 + 4 = 99 + 27 = 126$.

> **Investigate further:** Find the sum of the digits of $10^{100} - 2025$.

7. Find all pairs of positive integers (x, y) such that $x^3 - y^3 = 37$. *(after SMC 2023 Q1)*

Answer: $(4, 3)$

Method: Factorise the left side: $(x - y)(x^2 + xy + y^2) = 37$. Since 37 is prime and x, y are positive integers, $x^2 + xy + y^2 > x - y > 0$. Thus $x - y = 1$ and $x^2 + xy + y^2 = 37$. Substituting $x = y + 1$ gives $(y + 1)^2 + (y + 1)y + y^2 = 37 \implies 3y^2 + 3y - 36 = 0 \implies y^2 + y - 12 = 0 \implies (y + 4)(y - 3) = 0$. Since $y > 0$, $y = 3$, which gives $x = 4$.

> **Investigate further:** This question builds upon algebraic identity usage similar to SMC 2023 Q1. Can you find all integer pairs for $x^3 - y^3 = 19$?

8. Find the smallest positive integer x such that $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 4$, and $x \equiv 3 \pmod 5$.

Answer: 58

Method: From $x \equiv 3 \pmod 5$ and $x \equiv 2 \pmod 4$, the last digit must be 8 (since it must end in 3 or 8, and must be even). So $x = 10k + 8$. Since $10k + 8 \equiv 2k \equiv 2 \pmod 4$, k must be odd. We test odd k: for $k = 1$, $x = 18 \equiv 0 \pmod 3$. For $k = 3$, $x = 38 \equiv 2 \pmod 3$. For $k = 5$, $x = 58 \equiv 1 \pmod 3$. Thus 58 is the smallest.

Investigate further: Can you solve this using the Chinese Remainder Theorem directly?

Section D Challenge (TMUA / SMC difficulty)

Insight over computation. Each question has a short, elegant solution — find it.

1. How many integer pairs (x, y) satisfy the equation $x^2 + xy + y^2 = 7$? *(after SMC 2023 Q10)*

Answer: 12

Method: Multiply by 4: $4x^2 + 4xy + 4y^2 = 28$, which rewrites to $(2x + y)^2 + 3y^2 = 28$. Since squares are non-negative, $3y^2 \leq 28$, so $y^2 \in \{0, 1, 4, 9\}$. If $y^2 = 0$, $(2x)^2 = 28$ (no integer solution). If $y^2 = 1$, $(2x \pm 1)^2 = 25 \implies 2x \pm 1 = \pm 5$, giving $x \in \{2, -3, 3, -2\}$ (4 pairs). If $y^2 = 4$, $(2x \pm 2)^2 = 16 \implies 2x \pm 2 = \pm 4$, giving $x \in \{1, -3, 3, -1\}$ (4 pairs). If $y^2 = 9$, $(2x \pm 3)^2 = 1 \implies 2x \pm 3 = \pm 1$, giving $x \in \{-1, -2, 2, 1\}$ (4 pairs). In total, $4 + 4 + 4 = 12$ pairs.

Investigate further: This question explores bounding Diophantine equations, a more advanced application of finding factors as seen in SMC 2023 Q10. Can you find the number of integer pairs for $x^2 - xy + y^2 = 7$?

2. For how many integers $n \in \{1, 2, \ldots, 100\}$ is n^n a perfect square?

Answer: 55

Method: If n is even, say $n = 2k$, then $n^n = (2k)^{2k} = ((2k)^k)^2$, which is always a perfect square. There are 50 even integers in the range. If n is odd, say $n = 2k + 1$, then $n^n = n^{2k+1} = n \cdot (n^k)^2$. For this to be a perfect square, n itself must be a perfect square. The odd perfect squares in the range are $1, 9, 25, 49, 81$, which gives 5 more values. Total is $50 + 5 = 55$.

Investigate further: For how many integers in the same range is n^n a perfect cube?

3. Find all pairs of positive integers (a, b) such that $a^2 - 4b^2 = 45$. *(after MAT 2021 Q1)*

Answer: $(7, 1), (9, 3), (23, 11)$

Method: Factorise as $(a - 2b)(a + 2b) = 45$. The divisors of 45 are $(1, 45), (3, 15), (5, 9)$. We set $a - 2b$ to the smaller divisor and $a + 2b$ to the larger. For $(1, 45)$, adding gives $2a = 46 \implies a = 23$, $2b = 22 \implies b = 11$. For $(3, 15)$, $2a = 18 \implies a = 9$, $2b = 12 \implies b = 3$. For $(5, 9)$, $2a = 14 \implies a = 7$, $2b = 4 \implies b = 1$. Thus the pairs are $(7, 1), (9, 3), (23, 11)$.

Investigate further: This mirrors the difference of squares logic tested implicitly in MAT 2021 Q1. What happens if you try to solve $a^2 - 4b^2 = 46$?

4. What is the largest prime factor of $2^{16} - 1$?

Answer: 257

Method: We factorise using difference of two squares repeatedly: $2^{16} - 1 = (2^8 + 1)(2^8 - 1) = (257)(2^4 + 1)(2^4 - 1) = (257)(17)(2^2 + 1)(2^2 - 1) = 257 \times 17 \times 5 \times 3 \times 1$. The largest number in this

factorisation is 257, which is a known Fermat prime.

> **Investigate further:** What is the largest prime factor of $2^{18} - 1$?

5. If p and q are prime numbers such that $p^2 - 2q^2 = 1$, find the value of $p + q$. *(after SMC 2023 Q17)*

> **Answer:** 5

Method: Rewrite as $p^2 - 1 = 2q^2$, which gives $(p-1)(p+1) = 2q^2$. If $p = 2$, $3 = 2q^2$, impossible. Thus p is an odd prime. Then $p-1$ and $p+1$ are consecutive even integers, so one is a multiple of 4 and the other is a multiple of 2. Their product is divisible by 8. Thus $2q^2$ is a multiple of 8, meaning q^2 is a multiple of 4, so q is even. The only even prime is $q = 2$. Substituting $q = 2$ gives $p^2 - 8 = 1 \implies p^2 = 9 \implies p = 3$. Both 3 and 2 are prime, so $p + q = 5$.

> **Investigate further:** This question plays with parity constraints on primes, much like SMC 2023 Q17 logic. Can you solve $p^2 - 3q^2 = 1$ for primes p, q?

Top 5 Patterns Today

1. **Prime Factorisation.** Knowing the prime factorisation of numbers is the key to counting divisors, finding GCDs, and determining properties of factorials.

2. **Difference of Two Squares.** $(x - y)(x + y)$ appears everywhere. It simplifies evaluations like $a^2 - b^2$, factors polynomials, and solves Diophantine equations.

3. **Completing the Rectangle/Square.** Equations like $xy + ax + by = c$ can always be factored by adding ab to both sides, transforming them into $(x + b)(y + a) = c + ab$.

4. **Modular Arithmetic Cycles.** Powers of numbers modulo n always eventually fall into repeating cycles. You can compute the last digits of large powers by observing these short cycles.

5. **Parity Arguments.** Often, testing even/odd cases or investigating divisibility by 2 and 4 can tightly restrict solutions to Diophantine equations, especially when primes are involved.

Common Traps to Avoid

1. **Forgetting 1 is not a prime.** When finding prime factors or solving equations with primes, remember that 2 is the smallest prime. 1 is just a unit.

2. **Expanding unnecessarily.** Never expand an expression when you can keep it in factored form. The factors hold the secrets to divisibility and GCD.

3. **Missing negative solutions in Diophantine equations.** When solving $x^2 - y^2 = d$, don't forget that negative factors multiply to positive results. Read carefully if the question restricts to positive integers.

4. **Brute forcing modular equations.** For equations like $3x \equiv 2 \pmod{11}$, don't just guess numbers randomly; strategically add multiples of the modulus to the right-hand side until it is divisible by the coefficient.

5. **Assuming a prime must be odd.** 2 is the only even prime, and it is frequently the key that unlocks number theory puzzles involving squares and parity.

"Speed comes from recognition, not from rushing. Every shortcut is a pattern you have internalised."

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