Daily Number Theory Drill #6 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: The Euclidean Step · Polynomial Division · GCD Logic.

Section A

1. Find the largest prime factor of 2025.

Answer: 5

Method: We have $2025 = 25 \times 81 = 5^2 \times 3^4$. The largest prime factor is 5.

> **Investigate further:** Is 2025 a perfect square? Yes, $45^2 = 2025$.

2. What is the last digit of 3^{2026}?

Answer: 9

Method: The last digits of powers of 3 cycle every 4: $3, 9, 7, 1$. Since $2026 \equiv 2 \pmod 4$, the last digit is the second in the cycle, which is 9.

> **Investigate further:** What is the last digit of 3^{2024}?

3. Find the greatest common divisor of 84 and 120.

Answer: 12

Method: Prime factorisation gives $84 = 2^2 \times 3 \times 7$ and $120 = 2^3 \times 3 \times 5$. The GCD is $2^2 \times 3 = 12$.

> **Investigate further:** Can you find their LCM using the GCD?

4. What is the remainder when 10^{10} is divided by 9?

Answer: 1

Method: Using modular arithmetic, $10 \equiv 1 \pmod 9$. Therefore, $10^{10} \equiv 1^{10} \equiv 1 \pmod 9$.

> **Investigate further:** What about division by 11? $10 \equiv -1 \pmod{11}$.

5. Find the missing digit d such that the five-digit number $47d53$ is divisible by 9.

Answer: 8

Method: A number is divisible by 9 if the sum of its digits is divisible by 9. The sum is $4+7+d+5+3 = 19 + d$. For $19 + d$ to be a multiple of 9 with d a single digit, $19 + d = 27$, so $d = 8$.

> **Investigate further:** What if the number had to be divisible by 11?

6. How many positive integer factors does 100 have?

Answer: 9

Method: Prime factorisation: $100 = 2^2 \times 5^2$. The number of factors is $(2+1)(2+1) = 9$.

> **Investigate further:** Can you verify by listing them out?

7. Which is larger: 2^{300} or 3^{200}?

Answer: 3^{200}

Method: We can rewrite the powers: $2^{300} = (2^3)^{100} = 8^{100}$, and $3^{200} = (3^2)^{100} = 9^{100}$. Clearly $9^{100} > 8^{100}$.

> **Investigate further:** Which is larger: 2^{50} or 3^{30}?

8. Evaluate $5^{2024} \pmod 4$.

Answer: 1

Method: Since $5 \equiv 1 \pmod 4$, we have $5^{2024} \equiv 1^{2024} \equiv 1 \pmod 4$.

> **Investigate further:** Evaluate $3^{2025} \pmod 4$.

9. Find the smallest positive integer n such that $12n$ is a perfect square.

Answer: 3

Method: We have $12 = 2^2 \times 3$. For $12n$ to be a perfect square, all prime factors must have even exponents. We need another factor of 3, so $n = 3$.

> **Investigate further:** What is the smallest n making it a perfect cube?

10. What is the sum of the prime factors of 210?

Answer: 17

Method: We factorise $210 = 21 \times 10 = 3 \times 7 \times 2 \times 5$. The sum of these prime factors is $2+3+5+7 = 17$.

> **Investigate further:** Are there any repeated prime factors in 210?

Section B

1. Find the number of trailing zeroes of 20!.

Answer: 4

Method: The number of trailing zeroes is determined by the number of factors of 5. This is $\lfloor 20/5 \rfloor = 4$.

> **Investigate further:** How many trailing zeroes in 100!?

2. Solve for x in $3x \equiv 4 \pmod 7$, with $0 < x < 7$.

Answer: 6

Method: Multiply by 5, since $5 \times 3 = 15 \equiv 1 \pmod 7$. Thus $x \equiv 5 \times 4 = 20 \equiv 6 \pmod 7$.

> **Investigate further:** Verify by plugging $x = 6$ back in: $18 \pmod 7 = 4$.

3. If p and q are prime numbers such that $p^2 - q^2 = 24$, find the value of $p + q$.

Answer: 12

Method: Factorising as a difference of squares: $(p-q)(p+q) = 24$. Since p and q are primes, their sum and difference must have the same parity. So $(p-q)$ and $(p+q)$ are both even. The even factor pairs of 24 are $(2, 12)$ and $(4, 6)$. If $p + q = 12, p - q = 2 \implies p = 7, q = 5$, both prime! $p + q = 12$.

> **Investigate further:** What if $p^2 - q^2 = 33$?

4. Find the last two digits of 5^{2025}.

Answer: 25

Method: Any power of 5 greater than 1 ends in 25. $5^2 = 25$, $5^3 = 125$, $5^4 = 625$. By induction, $5^n \equiv 25 \pmod{100}$ for all $n \geq 2$.

> **Investigate further:** What are the last two digits of 6^{2025}?

5. A number N leaves a remainder of 2 when divided by 3, and 3 when divided by 5. What is the smallest positive integer value of N?

Answer: 8

Method: We need $N \equiv 2 \pmod 3$ and $N \equiv 3 \pmod 5$. Testing numbers ending in 3 or 8: $3 \equiv 0 \pmod 3$, but $8 \equiv 2 \pmod 3$. So $N = 8$.

> **Investigate further:** Solve this systematically using the Chinese Remainder Theorem.

6. How many perfect squares divide 3600?

Answer: 12

Method: Prime factorisation: $3600 = 2^4 \times 3^2 \times 5^2$. A perfect square divisor must have even exponents: $2^{2a} \times 3^{2b} \times 5^{2c}$. There are 3 choices for $a \in \{0, 1, 2\}$, 2 for $b \in \{0, 1\}$, and 2 for $c \in \{0, 1\}$. $3 \times 2 \times 2 = 12$.

> **Investigate further:** How many divisors in total does 3600 have?

7. Find the highest power of 3 dividing $27^5 \times 9^4$.

Answer: 23

Method: Convert bases to 3: $27^5 = (3^3)^5 = 3^{15}$ and $9^4 = (3^2)^4 = 3^8$. Multiply them: $3^{15} \times 3^8 = 3^{23}$. The highest power is 23.

> **Investigate further:** What is the highest power of 3 dividing $27^5 + 9^4$?

8. What is the smallest positive integer with exactly 6 positive divisors?

Answer: 12

Method: To have 6 factors, the prime factorisation must be p^5 or p^2q^1. $p^5 \geq 2^5 = 32$. For p^2q^1, the smallest is $2^2 \times 3^1 = 12$.

> **Investigate further:** What is the smallest positive integer with exactly 8 divisors?

9. If x and y are positive integers such that $xy = 144$ and $\gcd(x, y) = 3$, find the maximum possible value of $x + y$.

Answer: 51

Method: Let $x = 3a$ and $y = 3b$ where $\gcd(a, b) = 1$. $9ab = 144 \implies ab = 16$. The coprime factor pairs of 16 are just $(1, 16)$. Thus $x = 3, y = 48$. The maximum sum is $3 + 48 = 51$.

> **Investigate further:** What if the GCD was 4?

10. Calculate the remainder when $1! + 2! + 3! + \cdots + 10!$ is divided by 5.

Answer: 3

Method: Any factorial from 5! onwards is a multiple of 5, so they leave a remainder of 0. We only need $1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33 \equiv 3 \pmod 5$.

> **Investigate further:** Calculate the remainder when the sum is divided by 12.

Section C

1. Find the number of pairs of positive integers (x, y) such that $xy + x + y = 23$.

Answer: 6

Method: We use Simon's Favorite Factoring Trick to factorise as $(x + 1)(y + 1) - 1 = 23 \implies (x+1)(y+1) = 24$. The factors of 24 are $1, 2, 3, 4, 6, 8, 12, 24$. Since $x, y \geq 1$, we must have $x+1, y+1 \geq 2$. Valid pairs for $(x + 1, y + 1)$ are $(2, 12), (3, 8), (4, 6), (6, 4), (8, 3), (12, 2)$, giving 6 pairs.

> **Investigate further:** This question builds upon the archetype of Simon's Favorite Factoring Trick found in the SMC, pushing students to count valid pairs after factorisation.

2. Find the sum of all prime numbers p such that $p^2 + 2$ is also prime.

Answer: 3

Method: If $p \neq 3$, then $p \equiv 1$ or $2 \pmod 3$. In either case, $p^2 \equiv 1 \pmod 3$, so $p^2 + 2 \equiv 3 \equiv 0 \pmod 3$. Thus $p^2 + 2$ is a multiple of 3 and strictly greater than 3, meaning it cannot be prime. The only solution is $p = 3$, giving $3^2 + 2 = 11$. The sum is 3.

> **Investigate further:** This plays on the SMC archetype of checking prime structures modulo 3.

3. How many 3-digit palindromes are divisible by 11?

Answer: 8

Method: A 3-digit palindrome has the form $aba = 101a + 10b$. Divisibility by 11 requires $2a - b \equiv 0 \pmod{11}$. Since $1 \leq a \leq 9$ and $0 \leq b \leq 9$, the possible values for $2a - b$ are 0 or 11. If $2a - b = 0 \implies b = 2a \implies (a, b) = (1, 2), (2, 4), (3, 6), (4, 8)$ (4 cases). If $2a - b = 11 \implies b = 2a - 11 \implies (a, b) = (6, 1), (7, 3), (8, 5), (9, 7)$ (4 cases). Total 8.

> **Investigate further:** This adapts an SMC logic puzzle involving place value equations and divisibility rules.

4. The product of three consecutive positive integers is equal to 336. Find the sum of these integers.

Answer: 21

Method: Let the integers be $n-1, n, n+1$. Their product is $n(n^2-1) = 336$. We know $n^3 \approx 336$, so $n \approx 7$. Testing $n = 7$: $6 \times 7 \times 8 = 336$. The integers are $6, 7, 8$, and their sum is 21.

> **Investigate further:** A classic SMC archetype linking algebra (consecutive terms) to arithmetic limits.

5. Let N be a 4-digit number. When its digits are reversed, the new number is $N + 1089$. If the sum of the digits of N is 12, what is the maximum possible value of N?

Answer: 5016

Method: Let $N = \overline{abcd}$. The reversed number is $\overline{dcba}$. Their difference is $999(d-a) + 90(c-b) = 1089$. This implies $d - a = 1$ and $c - b = 1$. We are given $a+b+c+d = 12 \implies 2a+2b+2 = 12 \implies a+b = 5$. To maximise N, we want a to be as large as possible. Max $a = 5 \implies b = 0 \implies c = 1, d = 6$. The number is 5016.

> **Investigate further:** A multi-step logic problem commonly seen in SMC requiring place value decomposition.

6. Find the sum of all positive integers x such that $\frac{x+11}{x-3}$ is an integer.

Answer: 39

Method: We rewrite the fraction using polynomial division: $1 + \frac{14}{x-3}$. This is an integer when $x-3$ divides 14. The divisors of 14 are $\pm1, \pm2, \pm7, \pm14$. Thus $x \in \{4, 5, 10, 17, 2, 1, -4, -11\}$. The positive ones are $4, 5, 10, 17, 2, 1$, which sum to 39.

> **Investigate further:** Synoptic link: Rational manipulation combined with Diophantine divisor rules.

7. Find the maximum possible value of $\gcd(7n+15, 3n+2)$, where n is a positive integer.

Answer: 31

Method: Using the Euclidean algorithm trick $\gcd(a, b) = \gcd(a, a-kb)$, we can eliminate n. We have $\gcd(7n+15, 3n+2) = \gcd(7n+15-2(3n+2), 3n+2) = \gcd(n+11, 3n+2) = \gcd(n+11, 3n+2-3(n+11)) = \gcd(n+11, -31)$. The greatest common divisor must divide 31. Since 31 is prime, the maximum possible value is 31. This is achievable, for example when $n = 20$, $\gcd(155, 62) = 31$.

> **Investigate further:** This question tests the Euclidean algorithm trick, requiring students to take linear combinations of the two terms to eliminate the variable n, a common SMC archetype.

8. Find the sum of all integers n such that $\frac{n^2+3n+5}{n-1}$ is an integer.

Answer: 6

Method: We use polynomial division to rewrite the fraction: $\frac{n^2+3n+5}{n-1} = \frac{(n-1)(n+4)+9}{n-1} = n + 4 + \frac{9}{n-1}$. For this to be an integer, $n-1$ must be a divisor of 9. The divisors of 9 are $\pm1, \pm3, \pm9$. Thus, $n-1 \in \{-9, -3, -1, 1, 3, 9\}$, which gives $n \in \{-8, -2, 0, 2, 4, 10\}$. The sum of these values is $-8-2+0+2+4+10 = 6$.

> **Investigate further:** This question requires finding when polynomial quotients are integers by separating the remainder, extending the basic rational manipulation archetype to quadratics.

Section D

1. Find all three-digit numbers abc such that $abc = a! + b! + c!$.

Answer: 145

Method: The maximum digit cannot be ≥ 7, since $7! = 5040 > 999$. It cannot be 6, because even if there's a 6, the number is ≥ 720, meaning $a \geq 7$, contradiction. Thus the max digit is 5. The maximum possible sum is $3 \times 5! = 360$, so $a \leq 3$, which means $a! \leq 6$. We need b or c to be 5 to reach three digits. If $a = 1$, testing $(1, 4, 5) \implies 1! + 4! + 5! = 1 + 24 + 120 = 145$. The unique solution is 145.

> **Investigate further:** A famous BMO-style bounding argument using factorial growth limits.

2. Let $d_n = \gcd(n^2 + 3, n + 1)$. Find the sum of all possible values of d_n as n ranges over all positive integers.

Answer: 7

Method: Using polynomial division or the Euclidean step, we can rewrite $n^2 + 3$ in terms of $n + 1$. We have $n^2+3 = (n+1)(n-1)+4$. Therefore, $d_n = \gcd((n+1)(n-1)+4, n+1) = \gcd(4, n+1)$. The possible values for the greatest common divisor of $n + 1$ and 4 are the positive divisors of 4, which are $1, 2$, and 4. All these are achievable: $n = 2 \implies \gcd(7, 3) = 1$, $n = 1 \implies \gcd(4, 2) = 2$, $n = 3 \implies \gcd(12, 4) = 4$. The sum of these possible values is $1 + 2 + 4 = 7$.

> **Investigate further:** This question perfectly tests the Euclidean algorithm trick and polynomial quotients together, combining algebraic manipulation with greatest common divisor logic.

3. Find the last three digits of 2025^{2025}.

Answer: 625

Method: We are working modulo 1000. Note that $2025 \equiv 25 \pmod{1000}$. Successive powers of 25 are $25^2 = 625$, $25^3 = 15625 \equiv 625 \pmod{1000}$. In fact, $25^k \equiv 625 \pmod{1000}$ for all $k \geq 2$. Thus $2025^{2025} \equiv 25^{2025} \equiv 625 \pmod{1000}$.

> **Investigate further:** A rapid cycle-finding problem using modular exponentiation.

4. For how many positive integers $n \leq 100$ is n^n a perfect square?

Answer: 55

Method: If n is even, $n^n = (n^{n/2})^2$, which is always a perfect square since $n/2$ is an integer. There are 50 even numbers ≤ 100. If n is odd, then $n^n = n(n^{(n-1)/2})^2$. For this to be a square, n itself must be a perfect square. The odd perfect squares up to 100 are $1^2, 3^2, 5^2, 7^2, 9^2 \implies 1, 9, 25, 49, 81$. There are 5 such numbers. Total $= 50 + 5 = 55$.

> **Investigate further:** Combinatorial number theory archetype: classifying parity for exponents.

5. Find the maximum value of $x + y + z$ given that x, y, z are positive integers satisfying $x \leq y \leq z$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

Answer: 11

Method: If $x = 1$, the sum exceeds 1. If $x = 3$, $1/y + 1/z = 2/3$. Since $x \leq y \leq z$, $y \geq 3$. If $y = 3$, $z = 3 \implies 3 + 3 + 3 = 9$. If $x = 2$, $1/y + 1/z = 1/2 \implies (y - 2)(z - 2) = 4$. The factor pairs of 4 are $(1, 4)$ and $(2, 2)$. $(1, 4) \implies y = 3, z = 6 \implies 2+3+6 = 11$. $(2, 2) \implies y = 4, z = 4 \implies 2+4+4 = 10$. The max sum is 11.

> **Investigate further:** Egyptian fractions bounding archetype, requiring systematically trying minimal denominators.