# Daily Number Theory Drill #5 — Answers & Investigations

*Sheet by David Akinyele-Aje*

| Section | Topic | Questions | Target time |
|---|---|---|---|
| A | Rapid Recognition | 10 | 2 min 30 sec |
| B | Manipulation Drills | 10 | 8 min |
| C | Substitution & Structure | 8 | 10 min |
| D | Challenge | 5 | 15 min |

*New toolkit today: Bounding Diophantine Equations · Inequality Logic · Factor Searches.*

## Section A Rapid Recognition (≤15 s each)

*Last digits, primes, and basic properties. Pure recognition.*

**1.** Find the last digit of $3^{2025}$.

**Answer:** 3

**Method:** The last digits of powers of 3 follow a cycle of four: 3, 9, 7, 1. Since $2025 = 4 \times 506 + 1$, we have $2025 \equiv 1 \pmod 4$. The last digit is the first in the cycle, which is 3.

> **Investigate further:** Can you determine the last digit of $7^{2025}$? Hint: Find the cycle length for powers of 7.

**2.** Factorise 1001 completely into primes.

**Answer:** $7 \times 11 \times 13$

**Method:** Notice that 1001 is divisible by 11 ($1-0+0-1=0$). Dividing by 11 gives 91, and $91 = 7 \times 13$. Thus, the prime factorisation is $7 \times 11 \times 13$.

> **Investigate further:** What is the prime factorisation of 1001001?

**3.** Find $\gcd(42, 105)$.

**Answer:** 21

**Method:** Factorise both numbers: $42 = 2 \times 3 \times 7$ and $105 = 3 \times 5 \times 7$. The common prime factors are 3 and 7, so $\gcd(42, 105) = 3 \times 7 = 21$.

> **Investigate further:** What is the lowest common multiple of 42 and 105?

**4.** Find the number of positive divisors of 120.

**Answer:** 16

**Method:** Find the prime factorisation of 120: $120 = 2^3 \times 3^1 \times 5^1$. The number of divisors is $(3+1)(1+1)(1+1) = 4 \times 2 \times 2 = 16$.

> **Investigate further:** What is the sum of the positive divisors of 120?

**5.** Evaluate $5^{10} \pmod 6$.

**Answer:** 1

**Method:** Note that $5 \equiv -1 \pmod 6$. Thus, $5^{10} \equiv (-1)^{10} = 1 \pmod 6$.

> **Investigate further:** What is $5^{11} \pmod 6$?

**6.** Find the smallest positive integer $x$ such that $2x \equiv 1 \pmod 5$.

**Answer:** 3

**Method:** Multiply both sides by 3 (the inverse of 2 mod 5). This gives $6x \equiv 3 \pmod 5$, and since $6 \equiv 1 \pmod 5$, we get $x \equiv 3 \pmod 5$. The smallest positive integer is 3.

> **Investigate further:** What is the inverse of 3 modulo 7?

**7.** Find the smallest integer $n > 1$ such that $n \equiv 1 \pmod 3$ and $n \equiv 1 \pmod 4$.

**Answer:** 13

**Method:** By the Chinese Remainder Theorem, since $\gcd(3, 4) = 1$, the simultaneous congruences $n \equiv 1 \pmod 3$ and $n \equiv 1 \pmod 4$ imply $n \equiv 1 \pmod{12}$. The smallest integer $n > 1$ is $1 + 12 = 13$.

> **Investigate further:** Find the smallest integer $n > 2$ such that $n \equiv 2 \pmod 3$ and $n \equiv 2 \pmod 4$.

**8.** Write the prime factorisation of 360.

**Answer:** $2^3 \times 3^2 \times 5$

**Method:** Break down 360 into $36 \times 10 = (2^2 \times 3^2) \times (2 \times 5) = 2^3 \times 3^2 \times 5$.

> **Investigate further:** What is the prime factorisation of 720?

**9.** Calculate $7^3 \pmod{10}$.

**Answer:** 3

**Method:** Work modulo 10: $7^2 = 49 \equiv 9 \pmod{10}$, so $7^3 \equiv 9 \times 7 = 63 \equiv 3 \pmod{10}$.

> **Investigate further:** What is the last digit of $7^4$?

**10.** Find the largest prime factor of 9999.

**Answer:** 101

**Method:** Factorise 9999 as $9 \times 1111$. Then $1111 = 11 \times 101$. Since 101 is prime, the prime factorisation is $3^2 \times 11 \times 101$, and the largest prime factor is 101.

> **Investigate further:** What is the largest prime factor of 999999?

## Section B Manipulation Drills (≤50 s each)

*Modular arithmetic, exponentiation, divisibility rules. Look for patterns.*

**1.** The greatest power of 7 which is a factor of 50! is $7^k$. What is $k$? *(after SMC 2023 Q12)*

**Answer:** 8

**Method:** By Legendre's formula, the highest power of 7 dividing 50! is $\lfloor \frac{50}{7} \rfloor + \lfloor \frac{50}{49} \rfloor = 7 + 1 = 8$.

> **Investigate further:** This question scales an SMC concept. How would you calculate the number of trailing zeroes of 50!?

**2.** Find the smallest positive integer $n$ such that $15n$ is a perfect cube.

**Answer:** 225

**Method:** We have $15n = 3^1 \times 5^1 \times n$. To make this a perfect cube, the exponents of all prime factors must be multiples of 3. We need at least $3^2$ and $5^2$, so $n = 3^2 \times 5^2 = 9 \times 25 = 225$.

> **Investigate further:** What is the smallest $n$ such that $15n$ is a perfect square?

**3.** Find the remainder when $1! + 2! + 3! + \cdots + 100!$ is divided by 12.

**Answer:** 9

**Method:** Observe that for $k \geq 4$, $k!$ is a multiple of $4 \times 3 = 12$, so $k! \equiv 0 \pmod{12}$. The sum reduces to $1! + 2! + 3! = 1 + 2 + 6 = 9 \pmod{12}$.

> **Investigate further:** Find the remainder when the sum is divided by 24.

**4.** Which is larger, $3^{40}$ or $4^{30}$?

**Answer:** $3^{40}$

**Method:** Express both with the same exponent: $3^{40} = (3^4)^{10} = 81^{10}$ and $4^{30} = (4^3)^{10} = 64^{10}$. Since $81 > 64$, it follows that $3^{40} > 4^{30}$.

> **Investigate further:** Which is larger: $2^{50}$ or $3^{30}$?

**5.** Find the smallest positive integer $x$ such that $3x \equiv 4 \pmod{7}$.

**Answer:** 6

**Method:** Multiply both sides by 5 (the inverse of 3 mod 7, since $15 \equiv 1 \pmod{7}$). This gives $15x \equiv 20 \pmod{7}$, which simplifies to $x \equiv 6 \pmod{7}$.

> **Investigate further:** Solve $4x \equiv 5 \pmod{7}$.

**6.** Find the last two digits of $7^{2022}$.

**Answer:** 49

**Method:** By Euler's Totient Theorem, $\phi(100) = 40$, but we also know $7^4 = 2401 \equiv 1 \pmod{100}$. Since $2022 = 4 \times 505 + 2$, we have $7^{2022} \equiv 7^2 = 49 \pmod{100}$.

> **Investigate further:** What are the last two digits of $3^{2022}$?

**7.** How many positive factors of 1000 are perfect squares?

**Answer:** 4

**Method:** The prime factorisation of 1000 is $2^3 \times 5^3$. A perfect square factor must be of the form $2^{2a} \times 5^{2b}$. The possible values for $2a$ are 0 and 2, and for $2b$ are 0 and 2. There are $2 \times 2 = 4$ such factors.

> **Investigate further:** How many positive factors of 1000 are perfect cubes?

**8.** Find all integers $x$ such that $0 \leq x < 8$ and $x^2 \equiv 1 \pmod 8$.

**Answer:** $1, 3, 5, 7$

**Method:** Test the odd numbers mod 8: $1^2 = 1 \equiv 1$, $3^2 = 9 \equiv 1$, $5^2 = 25 \equiv 1$, $7^2 = 49 \equiv 1$. All four odd residues satisfy the congruence.

> **Investigate further:** Solve $x^2 \equiv 1 \pmod{16}$.

**9.** Given $2^x 3^y = 12^3$, find $x + y$.

**Answer:** 9

**Method:** Expand the right hand side: $12^3 = (2^2 \times 3)^3 = 2^6 \times 3^3$. By comparing exponents, $x = 6$ and $y = 3$. Hence $x + y = 9$.

> **Investigate further:** If $2^x 5^y = 100^2$, what is $x + y$?

**10.** Find all pairs of positive integers $(x, y)$ such that $xy = x + y + 3$.

**Answer:** $(2, 5), (3, 3), (5, 2)$

**Method:** Rearrange to $xy - x - y = 3$. Add 1 to both sides to complete the rectangle: $xy - x - y + 1 = 4$, which factors as $(x - 1)(y - 1) = 4$. Since $x, y \geq 1$, the pairs for $(x - 1, y - 1)$ are $(1, 4), (2, 2), (4, 1)$. Thus $(x, y)$ can be $(2, 5), (3, 3), (5, 2)$.

> **Investigate further:** Find the integer solutions to $xy = x + y + 5$.

## Section C Substitution & Structure (≤90 s each)

*Synoptic links: use algebraic identities and combinatorial counting to solve these number theory problems.*

**1.** The expression $\frac{n^2+3n+5}{n+1}$ takes integer values for certain positive integer values of $n$. What is the sum of all such positive integer values of $n$? *(after SMC 2023 Q19)*

**Answer:** 2

**Method:** Perform polynomial division: $\frac{n^2+3n+5}{n+1} = \frac{n(n+1)+2(n+1)+3}{n+1} = n + 2 + \frac{3}{n+1}$. For this to be an integer, $n + 1$ must divide 3. The positive divisors of 3 are 1 and 3, yielding $n = 0$ and $n = 2$. The only positive integer is $n = 2$, so the sum is 2.

> **Investigate further:** This adapts a core SMC structural problem. How would the answer change if $n$ could be any integer?

**2.** How many of the numbers 6, 7, 8, 9, 10 are factors of the sum $2^{2024} + 2^{2023} + 2^{2022}$? *(after SMC 2023 Q10)*

**Answer:** 2

**Method:** Factor out the lowest power: $2^{2022}(2^2 + 2^1 + 1) = 7 \times 2^{2022}$. Examining the options: 7 divides it, 8 divides it (since $2^3$ divides $2^{2022}$), but 6, 9, and 10 do not since they require a factor of 3 or 5.

> **Investigate further:** This questions mirrors SMC factorisation themes. Is 14 a factor?

**3.** Find all pairs of positive integers $(x, y)$ such that $x^2 - y^2 = 17$.

**Answer:** $(9, 8)$

**Method:** Factorise the difference of squares: $(x-y)(x+y) = 17$. Since 17 is prime and $x+y > x-y > 0$, we must have $x - y = 1$ and $x + y = 17$. Adding gives $2x = 18 \implies x = 9$, and $y = 8$.

> **Investigate further:** How many solutions exist if 17 is replaced by 15?

**4.** How many ordered pairs of positive integers $(a, b)$ satisfy $\text{lcm}(a, b) = 300$?

**Answer:** 75

**Method:** Prime factorise 300: $300 = 2^2 \times 3^1 \times 5^2$. Let $a = 2^{x_1}3^{y_1}5^{z_1}$ and $b = 2^{x_2}3^{y_2}5^{z_2}$. We need $\max(x_1, x_2) = 2$, $\max(y_1, y_2) = 1$, and $\max(z_1, z_2) = 2$. The number of pairs for an exponent $k$ is $2k + 1$. So $(2(2) + 1) \times (2(1) + 1) \times (2(2) + 1) = 5 \times 3 \times 5 = 75$.

> **Investigate further:** How many ordered pairs satisfy $\text{lcm}(a, b) = 1000$?

**5.** How many positive integers $n \leq 100$ have exactly 3 positive divisors?

**Answer:** 4

**Method:** A number has exactly 3 positive divisors if and only if it is the square of a prime. The primes whose squares are $\leq 100$ are 2, 3, 5, 7. Thus, there are 4 such numbers.

> **Investigate further:** How many numbers $\leq 100$ have exactly 4 positive divisors?

**6.** Find the sum of the digits of the largest integer $n$ such that $n^3 + 100$ is a multiple of $n + 10$.

**Answer:** 17

**Method:** Use polynomial division: $n^3 + 100 = (n^3 + 1000) - 900 = (n + 10)(n^2 - 10n + 100) - 900$. Thus, $n + 10$ must divide $-900$. The largest integer $n$ corresponds to the largest positive factor of 900, which is 900. Hence $n + 10 = 900 \implies n = 890$. The sum of the digits is $8 + 9 + 0 = 17$.

> **Investigate further:** Find the smallest integer $n$ satisfying this condition.

**7.** Find all triples of positive integers $(x, y, z)$ such that $x \leq y \leq z$ and $xyz = x + y + z$.

**Answer:** $(1, 2, 3)$

**Method:** Since $x \leq y \leq z$, we have $x + y + z \leq 3z$. Therefore $xyz \leq 3z \implies xy \leq 3$. Since $x$ and $y$ are positive integers with $x \leq y$, the possible pairs for $(x, y)$ are $(1, 1), (1, 2)$, and $(1, 3)$. If $(1, 1)$, $z = 2 + z$ (impossible). If $(1, 2)$, $2z = 3 + z \implies z = 3$. If $(1, 3)$, $3z = 4 + z \implies z = 2$, but we need $y \leq z$. The only solution is $(1, 2, 3)$.

> **Investigate further:** This builds upon BMO1 Diophantine constraints. Can you find all positive integer solutions $(x, y, z, w)$ to $xyzw = x + y + z + w$ with $x \leq y \leq z \leq w$?

**8.** Find the sum of all integer values of $n$ for which $n^2 + 4n + 3$ is a perfect square. *(after AI Prompts D5)*

**Answer:** -4

**Method:** Set $n^2 + 4n + 3 = k^2$. Complete the square: $(n + 2)^2 - 1 = k^2 \implies (n + 2)^2 - k^2 = 1$. The difference of two squares is 1 only when they are 1 and 0, so $(n+2)^2 = 1 \implies n+2 = \pm 1 \implies n = -1, -3$. The sum is $(-1) + (-3) = -4$.

> **Investigate further:** This expands an algebraic pattern into Diophantine analysis. Does $n^2+6n+5$ have similar properties?

## Section D Challenge (TMUA / SMC difficulty)

*Insight over computation. Each question has a short, elegant solution — find it.*

**1.** Find all prime numbers $p$ such that $p^2+2$ is also a prime number.

**Answer:** 3

**Method:** If $p=3$, $3^2+2=11$, which is prime. For $p>3$, $p \equiv 1$ or $2 \pmod 3$, which means $p^2 \equiv 1 \pmod 3$. Then $p^2+2 \equiv 3 \equiv 0 \pmod 3$. Since $p^2+2>3$, it must be a composite multiple of 3. So $p=3$ is the only solution.

> **Investigate further:** If $p^2+2$ is changed to $p^2+4$, are there any prime solutions?

**2.** Find all pairs of positive integers $(x,y)$ such that $x^2y = x^2+3y+2$.

**Answer:** $(2,6)$

**Method:** Rearrange to isolate $y$: $y(x^2-3) = x^2+2 \implies y = \frac{x^2+2}{x^2-3} = 1 + \frac{5}{x^2-3}$. For $y$ to be an integer, $x^2-3$ must be a factor of 5. The factors of 5 are $\pm1, \pm5$. Setting $x^2-3$ to $1, -1, 5, -5$ yields $x^2 = 4, 2, 8, -2$. The only perfect square is $x^2=4$, so $x=2$ (since $x>0$). This gives $y = 1+5/1 = 6$. The only pair is $(2,6)$.

> **Investigate further:** This explores algebraic rearrangement followed by a factor search. Find all integer pairs $(x,y)$ satisfying $x^2y = x^2+4y+3$.

**3.** How many pairs of positive integers $(x,y)$ with $x \le y$ satisfy $\frac{1}{x}+\frac{1}{y}=\frac{1}{6}$?

**Answer:** 5

**Method:** Since $x \le y$, $\frac{1}{y} \le \frac{1}{x}$. Thus $\frac{1}{6} = \frac{1}{x}+\frac{1}{y} \le \frac{2}{x}$, which implies $x \le 12$. We also need $\frac{1}{x} < \frac{1}{6}$, so $x > 6$. Testing $x \in \{7,8,9,10,11,12\}$ yields integer $y$ for $x = 7,8,9,10,12$. This gives 5 pairs: $(7,42), (8,24), (9,18), (10,15), (12,12)$.

> **Investigate further:** This question tests bounding Diophantine equations instead of using Simon's Favourite Factoring Trick. How many pairs $(x,y)$ with $x \le y$ satisfy $\frac{1}{x}+\frac{1}{y}=\frac{1}{p}$ for a prime $p$?

**4.** Find the remainder when $1^{2025}+2^{2025}+\cdots+10^{2025}$ is divided by 11.

**Answer:** 0

**Method:** By Fermat's Little Theorem, $x^{10} \equiv 1 \pmod{11}$ for $1 \le x \le 10$. Thus $x^{2025} = (x^{10})^{202} \times x^5 \equiv x^5 \pmod{11}$. Modulo 11, we have $(-x)^5 = -x^5 \equiv 11 - x^5$, so the terms pair up and cancel: $1^5+10^5 \equiv 1^5+(-1)^5 = 0 \pmod{11}$. This is true for all pairs, leaving a remainder of 0.

> **Investigate further:** Evaluate $1^{2026}+2^{2026}+\cdots+10^{2026} \pmod{11}$.

**5.** Find the sum of all integers $n$ for which $n^4+4$ is a prime number.

**Answer:** 0

**Method:** Apply the Sophie Germain Identity by adding and subtracting $4n^2$: $n^4+4 = (n^2+2)^2-(2n)^2 = (n^2-2n+2)(n^2+2n+2)$. For this product to be prime, the smaller factor must be 1 or -1. Since

$n^2 \pm 2n + 2 = (n \pm 1)^2 + 1 \geq 1$, we set $(n-1)^2 + 1 = 1 \implies n = 1$, yielding $1 \times 5 = 5$ (prime), and $(n+1)^2 + 1 = 1 \implies n = -1$, yielding $5 \times 1 = 5$ (prime). The sum of $n$ is $1 + (-1) = 0$.

> **Investigate further:** Use the identity to factorise $a^4 + 4b^4$.

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*"Number theory is the queen of mathematics." — Carl Friedrich Gauss*

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