Daily Number Theory Drill #4 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Prime Factor Algebra · Difference of Squares · Bounding Factors.

Section A

1. Find the largest prime factor of 1001.

Answer: 13

Method: $1001 = 7 \times 11 \times 13$. The largest prime factor is 13.

> **Investigate further:** Can you quickly recall the prime factors of 1001? Factoring three digit numbers that look prime but aren't is a standard trick.

2. What is the last digit of 2^{2026}?

Answer: 4

Method: The last digits of powers of 2 cycle in blocks of four: $2, 4, 8, 6$. The exponent 2026 leaves a remainder of 2 when divided by 4, so the last digit is the second in the cycle, which is 4.

> **Investigate further:** What is the last digit of 7^{2026}? Apply the same cyclic reasoning.

3. What is the remainder when $2^{10} + 3^{10}$ is divided by 5?

Answer: 3

Method: Modulo 5, we have $2^2 \equiv -1$ so $2^{10} \equiv (-1)^5 \equiv -1 \equiv 4$. Also, $3 \equiv -2$, so $3^{10} \equiv (-2)^{10} = 2^{10} \equiv 4$. The sum is $4 + 4 = 8 \equiv 3 \pmod 5$.

> **Investigate further:** How does this generalise for $a^n + b^n$ modulo p?

4. Find the greatest common divisor of 120 and 84.

Answer: 12

Method: Using the Euclidean algorithm: $120 = 84 \times 1 + 36$, $84 = 36 \times 2 + 12$, $36 = 12 \times 3 + 0$. The last non-zero remainder is 12.

> **Investigate further:** Use the prime factorisation method: $120 = 2^3 \times 3 \times 5$ and $84 = 2^2 \times 3 \times 7$. The minimum powers give $2^2 \times 3 = 12$.

5. How many positive integer factors does 144 have?

Answer: 15

Method: Prime factorisation: $144 = 12^2 = (2^2 \times 3)^2 = 2^4 \times 3^2$. The number of factors is $(4+1)(2+1) =$

$5 \times 3 = 15$.

> **Investigate further:** What is the product of all positive factors of 144?

6. What is the remainder when $11 \times 12 \times 13$ is divided by 10?

Answer: 6

Method: Compute modulo 10: $1 \times 2 \times 3 = 6$. So the remainder is 6.

> **Investigate further:** Can you apply modulo arithmetic to sums as easily as to products?

7. What is the smallest positive integer divisible by $2, 3, 4, 5$, and 6?

Answer: 60

Method: Find the least common multiple: $\text{lcm}(2, 3, 4, 5, 6) = \text{lcm}(4, 5, 6) = 60$.

> **Investigate further:** What is the smallest positive integer leaving a remainder of 1 when divided by $2, 3, 4, 5$, and 6?

8. Find the sum of the prime factors of 210.

Answer: 17

Method: Prime factorisation: $210 = 2 \times 3 \times 5 \times 7$. The sum of these prime factors is $2 + 3 + 5 + 7 = 17$.

> **Investigate further:** Can you do this for 330?

9. What is the remainder when $10^{2026} - 1$ is divided by 9?

Answer: 0

Method: $10 \equiv 1 \pmod 9$, so $10^{2026} \equiv 1^{2026} \equiv 1 \pmod 9$. Thus $10^{2026} - 1 \equiv 1 - 1 \equiv 0 \pmod 9$.

> **Investigate further:** This is the classic divisibility rule for 9. A number consisting of 2026 nines is a multiple of 9.

10. Find the value of k if 2^k is the highest power of 2 dividing 20!.

Answer: 18

Method: By Legendre's formula, the exponent is $\lfloor \frac{20}{2} \rfloor + \lfloor \frac{20}{4} \rfloor + \lfloor \frac{20}{8} \rfloor + \lfloor \frac{20}{16} \rfloor = 10 + 5 + 2 + 1 = 18$.

> **Investigate further:** Compute the highest power of 3 dividing 20!.

Section B

1. Find the smallest positive integer n such that $15n$ is a perfect square.

Answer: 15

Method: $15 = 3 \times 5$. To make $15n$ a perfect square, all prime exponents must be even. Multiply by another 3×5, so $n = 15$.

> **Investigate further:** What if we wanted $15n$ to be a perfect cube?

2. Find all single digits d such that the four-digit number $5d72$ is divisible by 9.

Answer: 4

Method: The sum of the digits must be a multiple of 9. $5 + d + 7 + 2 = d + 14$. The next multiple of 9 is 18, requiring $d = 4$. 27 requires $d = 13$, which isn't a digit.

> **Investigate further:** What digit makes it divisible by 11?

3. Find the number of trailing zeros in 50!.

Answer: 12

Method: Count the number of factors of 5 in 50! using Legendre's formula: $\lfloor \frac{50}{5} \rfloor + \lfloor \frac{50}{25} \rfloor = 10 + 2 = 12$.

> **Investigate further:** Can a factorial end in exactly 5 trailing zeros?

4. What is the remainder when 5^{2025} is divided by 7?

Answer: 6

Method: Modulo 7: $5 \equiv -2$. $5^2 \equiv (-2)^2 = 4$. $5^3 \equiv -8 \equiv -1 \equiv 6$. $5^6 \equiv (-1)^2 \equiv 1 \pmod 7$. The exponent $2025 = 6 \times 337 + 3$. Thus $5^{2025} \equiv (5^6)^{337} \times 5^3 \equiv 1 \times 5^3 \equiv -1 \equiv 6 \pmod 7$.

> **Investigate further:** This implicitly uses Fermat's Little Theorem! Notice how $a^{p-1} \equiv 1 \pmod p$ allows exponent reduction.

5. If $x \equiv 3 \pmod 7$, find the remainder when $x^2 + 4x + 5$ is divided by 7.

Answer: 5

Method: Substitute $x = 3$: $3^2 + 4(3) + 5 = 9 + 12 + 5 = 26$. Now $26 = 7 \times 3 + 5$, so the remainder is 5.

> **Investigate further:** Can you evaluate $(x + 2)^2 + 1 \pmod 7$ directly? $(3 + 2)^2 + 1 = 26 \equiv 5$.

6. Find the smallest positive integer that leaves a remainder of 2 when divided by 3, and 3 when divided by 5.

Answer: 8

Method: The numbers leaving remainder 3 when divided by 5 are $3, 8, 13, \ldots$. Checking divisibility by 3: $3 \equiv 0$, $8 \equiv 2$. So 8 is the smallest.

> **Investigate further:** This is a manual application of the Chinese Remainder Theorem. Can you solve it algebraically?

7. Find the sum of all positive integer divisors of 100.

Answer: 217

Method: $100 = 2^2 \times 5^2$. The sum of divisors is $(1 + 2 + 2^2)(1 + 5 + 5^2) = 7 \times 31 = 217$.

> **Investigate further:** What is the sum of the divisors of 1000?

8. What is the last digit of $3^{101} + 7^{101}$?

Answer: 0

Method: For 3: cycle is $3, 9, 7, 1$. $101 \equiv 1 \pmod 4$, so last digit is 3. For 7: cycle is $7, 9, 3, 1$. $101 \equiv 1 \pmod 4$, so last digit is 7. The sum $3 + 7 = 10$, so the last digit is 0.

> **Investigate further:** Can you see that $3^{101} + 7^{101}$ is divisible by 10 because 101 is odd and $a^n + b^n$ is divisible by $a + b$ when n is odd?

9. Find the number of positive divisors of $2^3 \times 3^4 \times 5$.

Answer: 40

Method: The exponents are $3, 4$, and 1. The number of divisors is $(3+1)(4+1)(1+1) = 4 \times 5 \times 2 = 40$.

> **Investigate further:** How many of these divisors are even?

10. Find the largest three-digit number that is a multiple of 11 and 13.

Answer: 858

Method: The $\text{lcm}(11, 13) = 143$. We want the largest multiple of 143 under 1000. $143 \times 10 = 1430$. $143 \times 6 = 858$, and $143 \times 7 = 1001$. Thus the answer is 858.

> **Investigate further:** Is there a shortcut to check divisibility by 143?

Section C

1. Find all pairs of positive integers (x, y) such that $x^2 - y^2 = 45$.

Answer: $(23, 22), (9, 6), (7, 2)$

Method: Factorise as $(x - y)(x + y) = 45$. Since x and y are positive integers, $x + y > x - y > 0$, and both must have the same parity (but 45 is odd, so one is even, one is odd). The factor pairs of 45 are $(1, 45), (3, 15), (5, 9)$. This gives three systems of equations. For $(1, 45)$, $2x = 46 \implies x = 23, y = 22$. For $(3, 15)$, $2x = 18 \implies x = 9, y = 6$. For $(5, 9)$, $2x = 14 \implies x = 7, y = 2$.

> **Investigate further:** This is based on the archetype in SMC 2018 Q15, requiring us to explicitly decompose a number as a product of integer factors representing sum and difference.

2. The prime number p can be expressed as $p = a^2 - b^2$, where a and b are also prime numbers. Find the value of p.

Answer: 5

Method: Factorise as $p = (a - b)(a + b)$. Since p is a prime number and a, b are positive integers, the smaller factor $a - b$ must equal 1. Thus, a and b are consecutive integers. The only consecutive integers that are both prime are 3 and 2. Therefore, $a = 3, b = 2$, and $p = 3^2 - 2^2 = 9 - 4 = 5$.

> **Investigate further:** This expands upon the archetype in SMC 2011 Q19, asking students to realise that difference of squares for a prime number implicitly forces a difference of 1, restricting the variables significantly.

3. Find the largest integer n such that $n^3 + 100$ is divisible by $n + 10$.

Answer: 890

Method: Use polynomial division or algebraic manipulation: $n^3 + 100 = (n^3 + 10^3) - 900 = (n + 10)(n^2 - 10n + 100) - 900$. Thus, $n + 10$ must divide 900. To maximise n, we maximise $n + 10$. The largest divisor of 900 is 900, so $n + 10 = 900 \implies n = 890$.

> **Investigate further:** Inspired by MAT 2016 Q1H, where the structure of algebraic identities simplifies divisibility problems involving polynomials.

4. How many ordered pairs of positive integers (x, y) satisfy $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$?

Answer: 9

Method: Multiply by $6xy$ to clear denominators: $6y + 6x = xy$. Rearrange and add 36 to both sides to complete the rectangle: $xy - 6x - 6y + 36 = 36$, which factors as $(x - 6)(y - 6) = 36$. The number of positive integer solutions corresponds to the number of positive divisors of 36. Since $36 = 2^2 \times 3^2$, there are $(2 + 1)(2 + 1) = 9$ divisors.

> **Investigate further:** A classic algebraic manipulation trick (Simon's Favorite Factoring Trick) drawn from SMC 2015 Q21, merging algebra and number theory counting.

5. Find all integer values of x for which $\frac{x^2+5x+14}{x+3}$ is an integer.

Answer: $-11, -7, -5, -4, -2, -1, 1, 5$

Method: Rewrite the numerator to isolate a multiple of $(x + 3)$: $x^2 + 5x + 14 = (x + 3)(x + 2) + 8$. So the expression is $x + 2 + \frac{8}{x+3}$. For this to be an integer, $x + 3$ must be a divisor of 8. The divisors of 8 are $\pm 1, \pm 2, \pm 4, \pm 8$. Setting $x + 3$ to each gives $x \in \{-11, -7, -5, -4, -2, -1, 1, 5\}$.

> **Investigate further:** Adapted from SMC 2019 Q20, highlighting the power of partial fractions and remainder theorem concepts over integers.

6. How many divisors of 10^{10} are perfect squares?

Answer: 36

Method: The prime factorisation of 10^{10} is $2^{10} \times 5^{10}$. A divisor of 10^{10} has the form $2^a 5^b$, where $0 \leq a, b \leq 10$. For it to be a perfect square, both a and b must be even. The even integers in this range are $0, 2, 4, 6, 8, 10$. There are 6 choices for a and 6 choices for b, giving $6 \times 6 = 36$ total combinations.

> **Investigate further:** Building upon SMC 2021 Q22, this links fundamental theorem of arithmetic with basic combinatorics (the multiplication principle).

7. How many positive integers $n \leq 1000$ are not divisible by $2, 3$, or 5?

Answer: 266

Method: Use the Principle of Inclusion-Exclusion (PIE). Count multiples: $\lfloor 1000/2 \rfloor = 500$, $\lfloor 1000/3 \rfloor = 333$, $\lfloor 1000/5 \rfloor = 200$. Pairwise LCMs: $\lfloor 1000/6 \rfloor = 166$, $\lfloor 1000/10 \rfloor = 100$, $\lfloor 1000/15 \rfloor = 66$. Triple LCM: $\lfloor 1000/30 \rfloor = 33$. Total multiples: $500 + 333 + 200 - (166 + 100 + 66) + 33 = 734$. The numbers NOT divisible are $1000 - 734 = 266$.

> **Investigate further:** A standard counting strategy explored in SMC 2017 Q19. It reinforces the intersection logic of sets with prime multiples.

8. Given that p and q are prime numbers satisfying $p = q^2 - 36$, find the value of p.

Answer: 13

Method: Factorise the right hand side as $p = (q - 6)(q + 6)$. Because p is prime and the factors are integers with $q + 6 > q - 6$, the smaller factor must be 1. Thus, $q - 6 = 1$, which gives $q = 7$. Checking, $q = 7$ is indeed a prime. The value of p is then $7 + 6 = 13$, which is also prime.

> **Investigate further:** This is based on the archetype in SMC 2014 Q22, substituting polynomials with prime constraints to test whether a student can spot that $x^2 - y^2 = p$ necessitates $x - y = 1$.

Section D

1. Find all pairs of positive integers (m, n) such that $3^m - 2^n = 1$.

Answer: $(1, 1), (2, 3)$

Method: If $m = 1$, $3 - 2^n = 1 \implies 2^n = 2 \implies n = 1$. If $m = 2$, $9 - 2^n = 1 \implies 2^n = 8 \implies n = 3$. Assume $m \geq 3$. Taking modulo 4, $3^m \equiv 1 \pmod 4$ since $2^n \equiv 0 \pmod 4$ for $n \geq 2$. Thus m must be even, say $m = 2k$. Then $3^{2k} - 1 = 2^n \implies (3^k - 1)(3^k + 1) = 2^n$. The factors must be powers of 2, differing by 2. The only powers of 2 that differ by 2 are 2 and 4, meaning $3^k - 1 = 2 \implies 3^k = 3 \implies k = 1, m = 2$. The only solutions are $(1, 1)$ and $(2, 3)$.

> **Investigate further:** This is a specific case of Catalan's Conjecture, echoing BMO1 2008 Q1. Considering remainders modulo 4 strongly restricts the parity of exponents.

2. For how many positive integers n is $n^4 + 4^n$ a prime number?

Answer: 1

Method: If $n = 1$, $1 + 4 = 5$, which is prime. If n is even and $n > 1$, $n^4 + 4^n$ is even and > 2, thus composite. If $n > 1$ is odd, let $n = 2k + 1$. Then $n^4 + 4^{2k+1} = n^4 + 4 \cdot (2^k)^4$. By Sophie Germain's Identity, $x^4 + 4y^4 = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$. Let $x = n, y = 2^k$. Both factors are strictly greater than 1, so the number is composite. Thus, there is exactly 1 solution.

> **Investigate further:** Inspired by BMO1 2013 Q2. Sophie Germain's Identity is an advanced but standard factorization trick that reveals hidden composites.

3. A sequence is defined by $a_1 = 1$, $a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$. What is the remainder when a_{2026} is divided by 3?

Answer: 1

Method: Compute the Fibonacci sequence modulo 3: $1, 1, 2, 0, 2, 2, 1, 0, 1, 1, \ldots$ The sequence of remainders repeats with a period of 8. We divide the index 2026 by 8: $2026 = 8 \times 253 + 2$. The remainder is the same as the 2nd term in the cycle, which is 1.

> **Investigate further:** Derived from SMC 2022 Q25. Taking sequences modulo m always produces periodic behavior by the Pigeonhole Principle.

4. Let p, q, and r be prime numbers such that $p = 9q^2 - r^2$. Find the value of p.

Answer: 11

Method: Factorise the expression as $p = (3q - r)(3q + r)$. Since p is a prime number, the smaller factor must be 1, so $3q - r = 1$, which implies $r = 3q - 1$. If $q > 2$, then q is an odd prime. This makes $3q - 1$ an even number strictly greater than 2, which means r cannot be prime. Thus, the only possible prime value for q is $q = 2$. Substituting this in gives $r = 3(2) - 1 = 5$, which is prime. The value of p is then $3(2) + 5 = 11$, which is also prime.

> **Investigate further:** This builds upon the archetype from BMO1 2010 Q1, adding parity arguments about prime numbers into a difference of squares problem, pushing the student to use exhaustive logical deduction.

5. Find the largest positive integer n such that 2^n divides $3^{1024} - 1$.

Answer: 12

Method: Using the factorization $x^2 - y^2 = (x - y)(x + y)$ repeatedly, $3^{1024} - 1 = (3^{512} + 1)(3^{256} + 1) \cdots (3^2 + 1)(3 + 1)(3 - 1)$. Every factor of the form $3^k + 1$ for even $k \geq 2$ is $3^{2m} + 1 \equiv 1 + 1 \equiv 2 \pmod 4$, meaning it contains exactly one factor of 2. There are 9 such factors ($k = 512, 256, \ldots, 2$). Thus we get 9 factors of 2. Additionally, $(3 + 1) = 4$ contains two factors of 2, and $(3 - 1) = 2$ contains one factor of 2. Total factors of 2: $1(\text{from } 3 - 1) + 2(\text{from } 3 + 1) + 9 \times 1(\text{from } 3^2 + 1 \text{ to } 3^{512} + 1) = 1 + 2 + 9 = 12$.

Investigate further: Mirrors BMO1 2019 Q3. This is an application of the Lifting The Exponent Lemma (LTE), $v_2(x^n - y^n) = v_2(x - y) + v_2(x + y) + v_2(n) - 1$.