# Daily Number Theory Drill #3 — Answers & Investigations

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| Section | Topic | Questions | Target time |
|---|---|---|---|
| A | Rapid Recognition | 10 | 2 min 30 sec |
| B | Manipulation Drills | 10 | 8 min |
| C | Substitution & Structure | 8 | 10 min |
| D | Challenge | 5 | 15 min |

*New toolkit today: Cyclic Patterns · Telescoping Exponents · Last Digit Sequences.*

**1.** Find the remainder when $10^{10} + 10^5 + 1$ is divided by 9.

**Answer:** 3

**Method:** Notice that $10 \equiv 1 \pmod 9$, so $10^{10} \equiv 1^{10} = 1 \pmod 9$. The sum modulo 9 is $1 + 1 + 1 = 3$.

> **Investigate further:** What is the remainder when $10^{100} + 1$ is divided by 11? Hint: $10 \equiv -1 \pmod{11}$.

**2.** What is the unit digit of $3^{2026}$?

**Answer:** 9

**Method:** The unit digits of powers of 3 cycle in the sequence $3, 9, 7, 1$. Since $2026 = 4 \times 506 + 2$, the digit corresponds to the 2nd position in the cycle, which is 9.

> **Investigate further:** What is the unit digit of $7^{2026}$?

**3.** Find the largest prime factor of 1001.

**Answer:** 13

**Method:** Recognise the classic product $1001 = 7 \times 11 \times 13$. The largest prime factor is therefore 13.

> **Investigate further:** Is 10001 prime? Hint: Check divisibility by 73 and 137.

**4.** Find the remainder when $7 \times 8 \times 9$ is divided by 5.

**Answer:** 4

**Method:** Compute modulo 5 on each factor: $7 \equiv 2$, $8 \equiv 3$, $9 \equiv 4$. Their product is $2 \times 3 \times 4 = 24$, which leaves a remainder of 4 when divided by 5.

> **Investigate further:** Find the remainder when $11 \times 12 \times 13$ is divided by 10.

**5.** If $n$ is an integer such that $2n \equiv 3 \pmod 5$, what is the smallest positive value of $n$?

**Answer:** 4

**Method:** Test small positive integers: $n = 1 \implies 2$, $n = 2 \implies 4$, $n = 3 \implies 6 \equiv 1$, $n = 4 \implies 8 \equiv 3$. Thus, the smallest is 4.

> **Investigate further:** Solve $3n \equiv 2 \pmod 7$ for the smallest positive $n$.

**6.** Evaluate $\gcd(144, 84)$.

**Answer:** 12

**Method:** Factor out the common multiple of 12: $\gcd(12 \times 12, 12 \times 7) = 12 \times \gcd(12, 7) = 12 \times 1 = 12$.

> **Investigate further:** What is the lowest common multiple of 144 and 84?

**7.** Find the number of positive divisors of 36.

**Answer:** 9

**Method:** Prime factorise as $36 = 2^2 \times 3^2$. The number of positive divisors is the product of one more than each exponent: $(2+1)(2+1) = 9$.

> **Investigate further:** How many positive divisors does 144 have?

**8.** Find the remainder when $2^{10}$ is divided by 11.

**Answer:** 1

**Method:** Note that $2^5 = 32 \equiv -1 \pmod{11}$. Squaring both sides gives $2^{10} \equiv (-1)^2 = 1 \pmod{11}$.

> **Investigate further:** What is $3^5 \pmod{11}$?

**9.** Express 120 as a product of prime factors.

**Answer:** $2^3 \times 3 \times 5$

**Method:** Break the number down as $10 \times 12 = (2 \times 5) \times (2^2 \times 3) = 2^3 \times 3 \times 5$.

> **Investigate further:** What is the prime factorisation of 360?

**10.** Find the smallest positive integer $k$ such that $12k$ is a perfect square.

**Answer:** 3

**Method:** Write $12k = 2^2 \times 3^1 \times k$. To make all exponents even, we need at least one more factor of 3, so $k = 3$.

> **Investigate further:** What is the smallest positive $k$ such that $12k$ is a perfect cube?

**11.** How many of the numbers $3, 4, 5, 6, 7$ are factors of the sum $3^{2026} + 3^{2025} + 3^{2024}$?

**Answer:** 1

**Method:** Factorise the expression as $3^{2024}(3^2 + 3^1 + 1) = 13 \times 3^{2024}$. The only prime factors are 3 and 13. Among the options given, only 3 divides it, giving 1 factor.

> **Investigate further:** How does this compare to SMC 2023 Q10? The archetype asks for factors of $2^{2024} + 2^{2023} + 2^{2022} = 7 \times 2^{2022}$.

**12.** Find the highest power of 2 that divides 100!.

**Answer:** 97

**Method:** Use Legendre's formula: $\lfloor 100/2 \rfloor + \lfloor 100/4 \rfloor + \lfloor 100/8 \rfloor + \cdots = 50 + 25 + 12 + 6 + 3 + 1 = 97$.

> **Investigate further:** Find the highest power of 5 that divides 100!.

**13.** If $a$ and $b$ are non-zero digits, find the maximum possible integer value of $\dfrac{10a+b}{a+b}$.

**Answer:** 9

**Method:** Let the ratio be $k$, so $10a + b = k(a + b)$, which rearranges to $9a = (k - 1)(a + b)$. To maximise $k$, we want $a + b$ to be as small as possible compared to $a$. Using $b = 1, a = 8$ yields $72 = (k - 1)(9) \implies k = 9$. Testing ratios quickly reveals $81/9 = 9$ is maximal.

> **Investigate further:** Can the ratio ever be exactly 10 if $b$ is allowed to be zero?

**14.** Find the number of integer pairs $(x, y)$ such that $xy = 24$.

**Answer:** 16

**Method:** Prime factorise $24 = 2^3 \times 3$. The number of positive divisors is $(3 + 1)(1 + 1) = 8$. For each positive divisor pair $(d, 24/d)$, there is also a corresponding negative pair $(-d, -24/d)$, giving $8 \times 2 = 16$ pairs.

> **Investigate further:** How many positive integer pairs $(x, y)$ satisfy $xy = 36$?

**15.** What is the smallest positive integer greater than 2 that leaves a remainder of 2 when divided by $3, 4,$ and $5$?

**Answer:** 62

**Method:** Let the number be $N$. Then $N - 2$ must be divisible by $3, 4,$ and $5$. The lowest common multiple of $3, 4, 5$ is 60. Therefore, the smallest positive $N$ is $60 + 2 = 62$.

> **Investigate further:** What if the remainders were 1, 2, and 3 when divided by 3, 4, and 5 respectively?

**16.** Find the remainder when $11^{11} + 12^{12}$ is divided by 10.

**Answer:** 7

**Method:** The unit digit of $11^{11}$ is 1. For $12^{12}$, we check the powers of 2: they cycle $2, 4, 8, 6$. Since 12 is a multiple of 4, the unit digit is 6. The sum modulo 10 is $1 + 6 = 7$.

> **Investigate further:** What is the remainder when $13^{13}$ is divided by 10?

**17.** Find the value of $p$ if $p$ and $p^2 + 2$ are both prime numbers.

**Answer:** 3

**Method:** Test $p = 2$: $4 + 2 = 6$ (not prime). Test $p = 3$: $9 + 2 = 11$ (prime). For any prime $p > 3$, $p$ cannot be divisible by 3, so $p \equiv \pm 1 \pmod 3 \implies p^2 \equiv 1 \pmod 3$. Then $p^2 + 2 \equiv 3 \equiv 0 \pmod 3$, so it is a multiple of 3 and greater than 3, meaning it is composite. Thus $p = 3$ is the only solution.

> **Investigate further:** Find $p$ if $p, p + 2,$ and $p + 4$ are all prime.

**18.** Given that $n$ is an integer, for how many values of $n$ is $\dfrac{2n+15}{n+2}$ an integer?

**Answer:** 4

**Method:** Rewrite the expression using algebraic long division: $\frac{2(n+2)+11}{n+2} = 2 + \frac{11}{n+2}$. For this to be an integer, $n + 2$ must be a divisor of 11. The integer divisors of 11 are $1, -1, 11, -11$, which gives 4 distinct values for $n$.

> **Investigate further:** Find the values of $n$ for which $\frac{n^2+1}{n+1}$ is an integer.

**19.** Find the sum of all positive integers $n$ for which $n^2 - 1$ is a prime number.

**Answer:** 2

**Method:** Factorise as $(n-1)(n+1)$. For the product of two positive integers to be prime, the smaller factor must be 1. Thus $n - 1 = 1$, yielding $n = 2$. Checking $n = 2$ gives $2^2 - 1 = 3$, which is prime. The sum of all valid $n$ is just 2.

> **Investigate further:** For what values of $n$ is $n^3 - 1$ a prime number?

**20.** Determine the two last digits of $5^{2026} + 6^{2026}$.

**Answer:** 81

**Method:** The last two digits of any power $5^k$ $(k \geq 2)$ are 25. For $6^k$, observe the pattern modulo 100: $36, 16, 96, 76, 56$, which repeats every 5 powers starting from $k = 2$. Since $2026 = 2 + 5 \times 404 + 4$, it corresponds to the 5th item in the cycle, which is 56. The sum is $25 + 56 = 81$.

> **Investigate further:** What are the last two digits of $7^{2026}$?

**21.** Find all pairs of positive integers $(x, y)$ such that $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{3}$.

**Answer:** $(4, 12), (6, 6), (12, 4)$

**Method:** Clear the denominators to get $3x + 3y = xy$, which rearranges to $xy - 3x - 3y = 0$. Apply Simon's Favorite Factoring Trick by adding 9 to both sides: $(x - 3)(y - 3) = 9$. The positive factor pairs of 9 are $(1, 9), (3, 3), (9, 1)$, which yield the pairs $(4, 12), (6, 6), (12, 4)$.

> **Investigate further:** Solve $\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$ for an arbitrary prime $p$.

**22.** Solve the equation $x^2 - y^2 = 2025$ for positive integers $(x, y)$ such that $y$ is as small as possible.

**Answer:** $x = 51, y = 24$

**Method:** Factorise as $(x - y)(x + y) = 2025$. To minimise $y$, the factors must be as close as possible without being equal. Since $\sqrt{2025} = 45$, the closest distinct factor pair is $75 \times 27$. Solving $x + y = 75$ and $x - y = 27$ yields $2x = 102 \implies x = 51$ and $y = 24$.

> **Investigate further:** Minimise $y$ for $x^2 - y^2 = 2024$.

**23.** Find all pairs of integers $(x, y)$ that satisfy $xy + x + y = 14$.

**Answer:** $(0, 14), (14, 0), (-2, -16), (-16, -2), (2, 4), (4, 2), (-4, -6), (-6, -4)$

**Method:** Add 1 to both sides to complete the factorisation: $(x+1)(y+1) = 15$. Consider all integer factor pairs of 15 (both positive and negative): $(1, 15), (15, 1), (-1, -15), (-15, -1), (3, 5), (5, 3), (-3, -5), (-5, -3)$. Subtract 1 from both coordinates of each pair to find $(x, y)$.

> **Investigate further:** How does this link to the structural trick used in Number Theory Sheet C1?

**24.** Find all integers $x$ such that $x^4 + x^2 + 1$ is a prime number.

**Answer:** $1, -1$

**Method:** Factorise algebraically by completing the square: $x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2 = (x^2 + x + 1)(x^2 - x + 1)$. For the product to be prime, the smaller positive factor must be 1. If $x \geq 0$, $x^2 - x + 1 = 1 \implies x = 0$ or $x = 1$. If $x \leq 0$, $x^2 + x + 1 = 1 \implies x = 0$ or $x = -1$. Testing $x = 0$ yields 1 (not prime), while $x = 1$ and $x = -1$ both yield 3 (prime).

> **Investigate further:** Solve for when $x^4 + 4$ is prime using Sophie Germain's identity.

**25.** How many ways can we choose the digits $a$ and $b$ such that the 5-digit number $3a4b2$ is divisible by 36?

**Answer:** 6

**Method:** Divisibility by 36 requires divisibility by 4 and 9. For 4, the number formed by the last two digits, $b2$, must be a multiple of 4, meaning $b \in \{1, 3, 5, 7, 9\}$. For 9, the digit sum $a + b + 9$ must be a multiple of 9, meaning $a + b$ is either 9 or 18. Each choice of $b$ from $\{1, 3, 5, 7\}$ yields exactly one digit $a$, while $b = 9$ allows $a = 0$ and $a = 9$. This yields $1 + 1 + 1 + 1 + 2 = 6$ ways.

> **Investigate further:** How many ways if the number must be divisible by 99?

**26.** Find the number of 3-element subsets of $\{1, 2, \ldots, 10\}$ such that the sum of the elements is divisible by 3.

**Answer:** 42

**Method:** Classify the numbers modulo 3: there are three 0s, four 1s, and three 2s. A 3-element subset sums to a multiple of 3 if all elements have the same remainder, or all three are distinct. Summing the combinations $\binom{3}{3} + \binom{4}{3} + \binom{3}{3} + \left(\binom{3}{1} \times \binom{4}{1} \times \binom{3}{1}\right) = 1 + 4 + 1 + 36 = 42$.

> **Investigate further:** How does this compare to the MAT combinatorics archetype? The archetype frequently groups elements by modulo properties to count subsets elegantly.

**27.** Find the sum of all integer values of $x$ for which $\dfrac{x^2 + 7x + 2}{x + 1}$ is an integer.

**Answer:** $-6$

**Method:** Perform algebraic long division: $\frac{x^2+7x+2}{x+1} = \frac{(x+1)(x+6)-4}{x+1} = x + 6 - \frac{4}{x+1}$. The expression is an integer when $x + 1$ is a divisor of 4. The possible values for $x + 1$ are $1, -1, 2, -2, 4, -4$, which give $x \in \{0, -2, 1, -3, 3, -5\}$. Their sum is $-6$.

> **Investigate further:** What is the sum of the values of $x$ for which $\frac{2x^2-1}{x-2}$ is an integer?

**28.** Find the last two digits of $2019^{2025}$.

**Answer:** 99

**Method:** We need $2019^{2025} \pmod{100}$, which is equivalent to $19^{2025} \pmod{100}$. Use the binomial expansion of $(20 - 1)^{2025}$. All terms involving $20^k$ for $k \geq 2$ are multiples of 100, so they vanish. The only surviving terms are $\binom{2025}{1}(20)(-1)^{2024} + (-1)^{2025} = 2025 \times 20 - 1 = 40500 - 1 \equiv 00 - 1 \equiv 99 \pmod{100}$.

> **Investigate further:** How does this compare to SMC binomial last digits archetypes? The archetype leverages the binomial expansion of $(10x \pm 1)^n$ to rapidly compute terminal digits without evaluating large powers.

**29.** Find the number of pairs of positive integers $(a, b)$ with $a \leq b$ such that $\text{lcm}(a, b) = 2^3 \cdot 3^2 \cdot 5$.

**Answer:** 53

**Method:** Let $\text{lcm}(a, b) = p^x q^y r^z$. For each prime $p$, the maximum of the exponents in $a$ and $b$ must be exactly $x$. There are $2x + 1$ such valid pairs of exponents. Multiplying these independent choices gives $(2(3) + 1)(2(2) + 1)(2(1) + 1) = 105$ ordered pairs. To find the unordered pairs where $a \leq b$, we compute $\frac{105+1}{2} = 53$.

> **Investigate further:** How does this build upon the BMO1 combinatorics archetype? The archetype requires constructing combinatorial matrix tables to trace prime exponent options.

**30.** Find all pairs of prime numbers $(p, q)$ such that $p^2 - 2q^2 = 1$.

**Answer:** $(3, 2)$

**Method:** Rearrange to $2q^2 = (p - 1)(p + 1)$. Since $p$ must be an odd prime (if $p = 2$, $3 \neq 2q^2$), $p - 1$ and $p + 1$ are consecutive even integers. Their product must be a multiple of 8, so $2q^2$ is a multiple of 8, meaning $q$ is even. The only even prime is $q = 2$, yielding $p = 3$.

> **Investigate further:** How does this relate to the BMO1 Diophantine archetype? The archetype often forces equations into parity checks that uniquely identify $q = 2$ as a prime.

**31.** Find the number of integer solutions to $x^3 + 2y^3 = 4z^3$.

**Answer:** 1

**Method:** Use the method of infinite descent. Notice that $x^3$ must be even, so $x$ is even. Let $x = 2X$, which gives $8X^3 + 2y^3 = 4z^3 \implies y^3 + 4X^3 = 2z^3$. This forces $y$ to be even. Substituting $y = 2Y$ forces $z$ to be even. The only integer that can be infinitely divided by 2 is 0, so $(0, 0, 0)$ is the unique solution.

> **Investigate further:** How does this relate to the BMO1 infinite descent archetype? The archetype proves that no non-zero solutions exist by showing any hypothetical solution can be perpetually halved.

**32.** Find the last three digits of $7^{9999}$.

**Answer:** 143

**Method:** We require $7^{9999} \pmod{1000}$. Note that $7^4 = 2401 \equiv 1 + 2400 \pmod{1000}$. Using the binomial expansion, $(1 + 2400)^{2499} \equiv 1 + 2499 \times 2400 \pmod{1000}$, since $2400^2$ and higher powers are multiples of 1000. $1 + 2499 \times 2400 = 1 + (2500 - 1) \times 2400 \equiv 1 - 2400 \equiv 601 \pmod{1000}$. Finally, $7^{9999} = 7^3 \times (7^4)^{2499} \equiv 343 \times 601 = 343 \times (600 + 1) = 205800 + 343 \equiv 143 \pmod{1000}$.

> **Investigate further:** How does this compare to SMC massive exponents archetypes? It extends the binomial technique to three digits, requiring careful modulo tracking of the first-order binomial term.

**33.** Find the last three digits of $2025^{2026}$.

**Answer:** 625

**Method:** We require $2025^{2026} \pmod{1000}$. Since $2025 \equiv 25 \pmod{1000}$, we consider powers of 25 modulo 1000. $25^1 = 25$, $25^2 = 625$, $25^3 = 15625 \equiv 625 \pmod{1000}$. Because $25^2 \equiv 25^3 \pmod{1000}$, all subsequent powers remain 625 modulo 1000. Thus, $25^{2026} \equiv 625 \pmod{1000}$.

> **Investigate further:** How does this relate to the SMC cyclic endings archetype? The archetype forces the student to spot that the sequence of remainders stabilises immediately, eliminating the need for complex cycles or totients.