Daily Number Theory Drill #2 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Modular Arithmetic · Remainder Equations · Parity in Modulo.

Section A Rapid Recognition

1. Prime factorise 1001.

Answer: $7 \times 11 \times 13$

Method: Recognise that 1001 is divisible by 11 since $1 - 0 + 0 - 1 = 0$. $1001 = 11 \times 91$. Then note $91 = 7 \times 13$, giving the prime factorisation $7 \times 11 \times 13$.

> **Investigate further:** Can you prime factorise 10001?

2. What is the remainder when 10^5 is divided by 11?

Answer: 10

Method: Notice that $10 \equiv -1 \pmod{11}$. Thus $10^5 \equiv (-1)^5 \equiv -1 \equiv 10 \pmod{11}$.

> **Investigate further:** What is the remainder when 10^5 is divided by 7?

3. Find the last digit of 7^{100}.

Answer: 1

Method: The last digits of powers of 7 cycle in a block of four: $7, 9, 3, 1$. Since 100 is exactly divisible by 4, 7^{100} must end in the fourth digit of the cycle, which is 1.

> **Investigate further:** Find the last digit of 3^{100}.

4. Find the smallest positive integer k such that $12k$ is a perfect square.

Answer: 3

Method: Prime factorise $12 = 2^2 \times 3^1$. For a perfect square, all prime exponents must be even, so we must multiply by an additional 3^1 to get $2^2 \times 3^2$. Thus $k = 3$.

> **Investigate further:** What is the smallest positive integer k such that $24k$ is a perfect cube?

5. Find the greatest common divisor of 84 and 120.

Answer: 12

Method: Prime factorise $84 = 2^2 \times 3 \times 7$ and $120 = 2^3 \times 3 \times 5$. The minimum exponents give the GCD: $2^2 \times 3 = 12$. Alternatively, use the Euclidean algorithm.

> **Investigate further:** Find the greatest common divisor of 144 and 216.

6. What is the smallest prime $p > 30$?

Answer: 31

Method: Test numbers after 30. 31 is not divisible by $2, 3$, or 5. Since $\sqrt{31} < 6$, we only need to check primes up to 5. Thus 31 is prime.

> **Investigate further:** What is the smallest prime $p > 40$?

7. Find the remainder when 3^{10} is divided by 8.

Answer: 1

Method: Notice that $3^2 = 9 \equiv 1 \pmod 8$. Thus $3^{10} = (3^2)^5 \equiv 1^5 \equiv 1 \pmod 8$.

> **Investigate further:** What is the remainder when 5^{10} is divided by 8?

8. Find the sum of all prime factors of 210.

Answer: 17

Method: Prime factorise $210 = 10 \times 21 = 2 \times 5 \times 3 \times 7$. The distinct prime factors are $2, 3, 5, 7$, which sum to 17.

> **Investigate further:** Find the sum of all prime factors of 330.

9. For what single digit d is the number $53d2$ a multiple of 9?

Answer: 8

Method: A number is divisible by 9 if the sum of its digits is divisible by 9. The sum is $5+3+d+2 = 10+d$. The nearest multiple of 9 is 18, which gives $d = 8$.

> **Investigate further:** For what single digit d is $53d2$ a multiple of 11?

10. Find the largest integer k such that 3^k divides 27^5.

Answer: 15

Method: Express the base as a power of 3: $27 = 3^3$. Thus $27^5 = (3^3)^5 = 3^{15}$. The largest integer k is 15.

> **Investigate further:** Find the largest integer k such that 2^k divides 16^8.

Section B Manipulation Drills

1. Find the number of positive divisors of 144.

Answer: 15

Method: Prime factorise $144 = 12^2 = (2^2 \times 3)^2 = 2^4 \times 3^2$. The number of divisors is the product of the exponents plus one: $(4+1)(2+1) = 5 \times 3 = 15$.

> **Investigate further:** Find the number of positive divisors of 360.

2. Find the smallest positive integer $x > 1$ that leaves a remainder of 1 when divided by $3, 4$, and

5.

Answer: 61

Method: Since $x \equiv 1 \pmod{m}$ for $m \in \{3, 4, 5\}$, it follows that $x - 1$ is a multiple of 3, 4, and 5. The least common multiple is $\text{lcm}(3, 4, 5) = 60$, so $x - 1 = 60k$. The smallest $x > 1$ is 61.

Investigate further: Find the smallest integer $x > 1$ leaving a remainder of 2 when divided by 4, 5, and 6.

3. Solve for the smallest positive integer x such that $5x \equiv 1 \pmod{7}$.

Answer: 3

Method: We seek an x where $5x = 7k + 1$. Testing values or using the fact that $5 \times 3 = 15 \equiv 1 \pmod{7}$ yields $x = 3$.

Investigate further: Solve for the smallest positive integer x such that $3x \equiv 1 \pmod{8}$.

4. Find the highest power of 5 dividing 100!.

Answer: 24

Method: Use Legendre's formula: $\lfloor 100/5 \rfloor + \lfloor 100/25 \rfloor = 20 + 4 = 24$.

Investigate further: Find the highest power of 3 dividing 100!.

5. Find the last two digits of 51^2.

Answer: 01

Method: Use the binomial expansion on $(50 + 1)^2 = 2500 + 100 + 1 = 2601$. The last two digits are 01.

Investigate further: Find the last two digits of 49^2.

6. Find the smallest positive integer n such that $n!$ is a multiple of 100.

Answer: 10

Method: Since $100 = 2^2 \times 5^2$, we need the prime factorisation of $n!$ to contain at least two 5s. The multiples of 5 are 5 and 10, so 10! contains exactly two 5s and more than enough 2s.

Investigate further: Find the smallest positive integer n such that $n!$ is a multiple of 1000.

7. Given that a and b are distinct positive integers with $\gcd(a, b) = 12$ and $\text{lcm}(a, b) = 144$, find the minimum possible value of $a + b$.

Answer: 84

Method: We can write $a = 12x$ and $b = 12y$ with $\gcd(x, y) = 1$. Since $ab = \gcd(a, b)\text{lcm}(a, b) = 1728$, we have $144xy = 1728 \implies xy = 12$. The coprime pairs (x, y) are $(1, 12)$ and $(3, 4)$. The sums $a + b$ are $12(1 + 12) = 156$ and $12(3 + 4) = 84$. The minimum is 84.

Investigate further: If instead $\text{lcm}(a, b) = 360$, what is the minimum value of $a + b$?

8. What is the sum of the digits of the number $2^{10} \times 5^8$?

Answer: 4

Method: We can combine powers of 10: $2^{10} \times 5^8 = 2^2 \times (2 \times 5)^8 = 4 \times 10^8$. This is a 4 followed by eight zeros, so the sum of the digits is 4.

> **Investigate further:** What is the sum of the digits of $2^{2024} \times 5^{2025}$?

9. What is the remainder when $2^{20} + 3^{20}$ is divided by 5?

Answer: 2

Method: Using Fermat's Little Theorem or simply noting patterns: $2^4 = 16 \equiv 1 \pmod 5$ and $3^4 = 81 \equiv 1 \pmod 5$. Thus $2^{20} = (2^4)^5 \equiv 1 \pmod 5$ and $3^{20} = (3^4)^5 \equiv 1 \pmod 5$. The sum is $1 + 1 = 2 \pmod 5$.

> **Investigate further:** What is the remainder when $4^{20} + 6^{20}$ is divided by 5?

10. Find the number of trailing zeros in 125!.

Answer: 31

Method: The number of trailing zeros is given by the highest power of 5 dividing 125!. Legendre's formula gives $\lfloor 125/5 \rfloor + \lfloor 125/25 \rfloor + \lfloor 125/125 \rfloor = 25 + 5 + 1 = 31$.

> **Investigate further:** Find the number of trailing zeros in 250!.

Section C Substitution & Structure

1. Find all positive integers n such that $n^2 - 4n - 5$ is a prime number. *(after SMC 2021 Q14)*

Answer: 6

Method: Factorising gives $(n-5)(n+1)$. For this to be a prime p, the smaller factor must be 1 (since $n + 1 > 1$). Thus $n - 5 = 1 \implies n = 6$. Substituting $n = 6$ yields $1 \times 7 = 7$, which is indeed prime.

> **Investigate further:** The archetype asks about $n^2 - 1$ being prime. Here we disguised the factorisation by adding linear terms, requiring the solver to pair factors correctly.

2. How many integers x are there such that $\dfrac{x^2 + 11}{x - 1}$ is an integer? *(after SMC 2019 Q20)*

Answer: 12

Method: Rewrite the fraction using algebraic division: $\frac{x^2-1+12}{x-1} = x + 1 + \frac{12}{x-1}$. For this to be an integer, $x - 1$ must divide 12. The divisors of 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$, which gives 12 divisors and thus 12 values for x.

> **Investigate further:** The archetype involves manipulating rational expressions. Here we require division to isolate a remainder term over a linear factor.

3. Find the sum of all positive integers n such that $n^2 + 8n$ is a perfect square. *(after BMO1 2018 Q1)*

Answer: 1

Method: Complete the square: $n^2 + 8n = k^2 \implies (n+4)^2 - 16 = k^2 \implies (n + 4 - k)(n + 4 + k) = 16$. Since $n > 0$, the sum of factors $n + 4 + k > 4$. Both factors must have the same parity (even), leaving only the factor pair 2×8. Thus $n + 4 + k = 8$ and $n + 4 - k = 2$. Adding these gives $2n + 8 = 10 \implies n = 1$.

> **Investigate further:** This question builds on the BMO1 2018 archetype by directly testing the solver's ability to complete the square and analyse factor parity.

4. How many 3-digit numbers have the property that the product of their digits is a prime number? *(after SMC 2023 Q17)*

Answer: 12

Method: For the product of three digits to be a prime p, two digits must be 1 and the other must be p. The single-digit primes are $2, 3, 5, 7$. For each prime, there are 3 arrangements (e.g. $211, 121, 112$). Total $= 4 \times 3 = 12$.

> **Investigate further:** The archetype involves properties of digit strings. This synoptically links number theory (primes) with a simple combinatorial counting exercise.

5. How many integers x with $1 \leq x \leq 100$ satisfy $x^2 \equiv 4 \pmod 5$? *(after SMC 2018 Q16)*

Answer: 40

Method: The quadratic residues modulo 5 are $0^2 \equiv 0$, $1^2 \equiv 1$, $2^2 \equiv 4$, $3^2 \equiv 4$, $4^2 \equiv 1$. Thus $x^2 \equiv 4 \pmod 5$ implies $x \equiv 2$ or $3 \pmod 5$. There are 2 solutions in every block of 5 consecutive integers. In the range $1 \leq x \leq 100$, there are $100/5 = 20$ blocks, yielding exactly $20 \times 2 = 40$ solutions.

> **Investigate further:** This explores quadratic residues modulo 5. The SMC 2018 Q16 archetype tested modular cycles; our question explicitly isolates the remainder equation structure.

6. Find the number of ordered pairs of positive integers (x, y) such that $x+y = 100$ and $\gcd(x, y) = 5$. *(after SMC 2022 Q12)*

Answer: 8

Method: Let $x = 5a$ and $y = 5b$. Then $a + b = 20$ with $\gcd(a, b) = 1$. We need the number of integers $a < 20$ coprime to 20. Using Euler's totient function or manual counting, $\phi(20) = 20(1-1/2)(1-1/5) = 8$. (The coprimes are $1, 3, 7, 9, 11, 13, 17, 19$).

> **Investigate further:** This question uses the classic structure of counting coprime pairs from the SMC, blending greatest common divisors with combinatorial counting.

7. Find the largest integer n such that $n^3 + 100$ is divisible by $n + 10$. *(after MAT 2018 Q1)*

Answer: 890

Method: Rewrite the numerator using the sum of cubes: $n^3 + 100 = (n^3 + 10^3) - 900 = (n + 10)(n^2 - 10n + 100) - 900$. Thus $n + 10$ must divide 900. To maximise n, we set $n + 10 = 900 \implies n = 890$.

> **Investigate further:** The MAT archetype uses algebra to isolate a constant remainder. Here we disguise a sum of cubes, rewarding algebraic structure recognition.

8. Find the sum of all positive integers $x < 20$ such that $x^2 \equiv 5 \pmod{11}$. *(after SMC 2019 Q22)*

Answer: 44

Method: Testing squares modulo 11: $1^2 = 1, 2^2 = 4, 3^2 = 9, 4^2 = 16 \equiv 5 \pmod{11}$. Since 4 is a solution, $-4 \equiv 7 \pmod{11}$ is also a solution. The positive integers $x < 20$ satisfying this are $4, 7, 4 + 11 = 15$, and $7 + 11 = 18$. Their sum is $4 + 7 + 15 + 18 = 44$.

> **Investigate further:** Testing a concrete remainder equation. The SMC 2019 Q22 archetype involved squares in modular arithmetic, and we extend it to sum the explicit bounded solutions.

Section D Challenge

1. A 4-digit number A is formed by arranging the digits $1, 2, 3, 4$ in some order. What is the sum of all such possible numbers A? *(after SMC 2021 Q21)*

Answer: 66660

Method: There are $4! = 24$ such numbers. Each digit $1, 2, 3, 4$ appears in each place value (units, tens, hundreds, thousands) exactly $3! = 6$ times. The sum of the digits is 10. Thus each place value contributes $6 \times 10 = 60$. The total sum is $60 \times 1111 = 66660$.

> **Investigate further:** This problem scales combinatorial permutations into a number theory context, requiring an understanding of base-10 expansions.

2. Find the number of pairs of positive integers (a, b) such that $a^2 - b^2 = 2024$. *(after BMO1 2021 Q1)*

Answer: 4

Method: Factor as $(a - b)(a + b) = 2024 = 2^3 \times 11 \times 23$. Since their sum is $2a$, $a - b$ and $a + b$ must share the same parity. Their product is even, so both must be even. Let $a - b = 2u$ and $a + b = 2v$. Then $4uv = 2024 \implies uv = 506 = 2 \times 11 \times 23$. The number 506 has 8 divisors. This yields exactly 4 pairs of (u, v) with $u < v$, giving 4 valid pairs for (a, b).

> **Investigate further:** This Diophantine equation is typical of BMO1 Q1. It evaluates factor pairing and strictly enforcing parity constraints.

3. Let S be the number of trailing zeros of 2025!. What is the remainder when S is divided by 100? *(after SMC 2023 Q12)*

Answer: 5

Method: Use Legendre's formula to find the exponent of 5: $\lfloor 2025/5 \rfloor + \lfloor 2025/25 \rfloor + \lfloor 2025/125 \rfloor + \lfloor 2025/625 \rfloor = 405 + 81 + 16 + 3 = 505$. So $S = 505$, and $505 \pmod{100} = 5$.

> **Investigate further:** Trailing zero questions are classic SMC. We ask for the remainder mod 100 to discourage partial manual additions.

4. How many pairs of positive integers (x, y) with $1 \leq x, y \leq 100$ satisfy $x^2 + y^2 \equiv 0 \pmod 5$? *(after BMO1 2017 Q2)*

Answer: 3600

Method: Modulo 5, the quadratic residues are $0, 1, 4$. For $x^2 + y^2 \equiv 0 \pmod 5$, the possible residue pairs (x^2, y^2) are $(0, 0), (1, 4), (4, 1)$. The number of solutions modulo 5 for $x^2 \equiv 0$ is 1; for $x^2 \equiv 1$ is 2; for $x^2 \equiv 4$ is 2. Hence, the number of pairs $(x, y) \pmod 5$ is $(1 \times 1) + (2 \times 2) + (2 \times 2) = 1 + 4 + 4 = 9$. Since there are exactly $100/5 = 20$ complete blocks of 5 for each variable, the total number of pairs is $9 \times 20 \times 20 = 3600$.

> **Investigate further:** This expands BMO1 2017 Q2's theme of Diophantine divisibility into a bivariate remainder equation, demanding combinatorial tracking of multiple modular branches.

5. Find the maximum value of $\gcd(n^2 + 3, n + 2)$ as n varies over all positive integers. *(after BMO1*

2019 Q2)

Answer: 7

Method: Use the Euclidean algorithm polynomial division logic: $n^2 + 3 = (n-2)(n+2) + 7$. Therefore $\gcd(n^2+3, n+2) = \gcd(7, n+2)$. The greatest common divisor can never exceed 7. It achieves 7 when $n + 2 = 7 \implies n = 5$.

> **Investigate further:** This relates a BMO1 polynomial GCD technique back to standard Euclidean manipulation. A short, elegant insight bounds the problem immediately.

"Speed comes from recognition, not from rushing. Every shortcut is a pattern you have internalised."

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