Daily Number Theory Drill #1 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

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Section A Rapid Recognition

1. Factorise completely into primes: 2025.

Answer: $3^4 \times 5^2$

Method: Recognise 2025 ends in 25, so it is divisible by 25. $2025 = 25 \times 81$. We know $25 = 5^2$ and $81 = 3^4$, giving $3^4 \times 5^2$.

> **Investigate further:** Can you quickly prime factorise 2024?

2. What is the remainder when 10^6 is divided by 7?

Answer: 1

Method: Notice that $10 \equiv 3 \pmod 7$. Thus $10^6 \equiv 3^6 \pmod 7$. We know $3^3 = 27 \equiv -1 \pmod 7$, so $3^6 = (-1)^2 = 1 \pmod 7$.

> **Investigate further:** What is the remainder when 10^6 is divided by 13?

3. Find the last digit of 3^{2024}.

Answer: 1

Method: The last digits of powers of 3 cycle in a block of four: $3, 9, 7, 1$. Since 2024 is exactly divisible by 4, 3^{2024} must end in the fourth digit of the cycle, which is 1.

> **Investigate further:** Find the last digit of 7^{2024}.

4. For what single digit d is the number $73d4$ divisible by 9?

Answer: 4

Method: A number is divisible by 9 if the sum of its digits is divisible by 9. The sum is $7+3+d+4 = 14+d$. The nearest multiple of 9 is 18, which gives $d = 4$.

> **Investigate further:** For what single digit d is $73d4$ divisible by 11?

5. The positive integers a and b are distinct. Given $\gcd(a, b) = 6$ and $\text{lcm}(a, b) = 36$, find $a + b$.

Answer: 42

Method: We know $ab = \gcd(a, b) \times \text{lcm}(a, b) = 6 \times 36 = 216$. Since both a and b must be multiples of 6, we can write $a = 6x$ and $b = 6y$ where $\gcd(x, y) = 1$ and $xy = 6$. The pairs for (x, y) are $(1, 6)$ and

$(2, 3)$. The pairs (a, b) are $(6, 36)$ and $(12, 18)$. The maximum possible sum is $6 + 36 = 42$.

> **Investigate further:** Find the minimum possible value of $a + b$ under the same conditions.

6. Find the largest integer k such that 2^k divides $20!$.

Answer: 18

Method: Use Legendre's Formula: count the multiples of $2, 4, 8, 16$ up to 20. We get $\lfloor 20/2 \rfloor + \lfloor 20/4 \rfloor + \lfloor 20/8 \rfloor + \lfloor 20/16 \rfloor = 10 + 5 + 2 + 1 = 18$.

> **Investigate further:** Find the largest integer k such that 3^k divides $20!$.

7. Find the remainder when 11^{10} is divided by 10.

Answer: 1

Method: Any number ending in 1 raised to any power will always end in 1. Thus, the last digit is 1, which is exactly the remainder when divided by 10.

> **Investigate further:** Find the remainder when 11^{10} is divided by 100.

8. Find the remainder when $2^{100} + 3^{100}$ is divided by 5.

Answer: 2

Method: By Fermat's Little Theorem (or simply cycling remainders), $2^4 \equiv 1 \pmod 5$ and $3^4 \equiv 1 \pmod 5$. Since 100 is a multiple of 4, $2^{100} \equiv 1$ and $3^{100} \equiv 1$. Summing them yields $1 + 1 = 2 \pmod 5$.

> **Investigate further:** Find the remainder when $2^{100} + 4^{100}$ is divided by 5.

9. What is the smallest prime factor of 1001?

Answer: 7

Method: Recognise 1001 as a famous product: $1001 = 7 \times 11 \times 13$. The smallest prime factor is 7.

> **Investigate further:** What is the prime factorisation of 1001001?

10. Find the only positive integer n such that $n^2 - 1$ is a prime number.

Answer: 2

Method: Factorise $n^2 - 1 = (n - 1)(n + 1)$. For this to be prime, the smaller factor $(n - 1)$ must be 1, giving $n = 2$. Checking this gives $n^2 - 1 = 3$, which is prime.

> **Investigate further:** Find the only prime number p such that $p^2 - 1$ is a prime.

Section B Manipulation Drills

1. Find the number of positive divisors of 360.

Answer: 24

Method: Prime factorise $360 = 36 \times 10 = 2^3 \times 3^2 \times 5^1$. Add one to each exponent and multiply: $(3 + 1)(2 + 1)(1 + 1) = 4 \times 3 \times 2 = 24$.

> **Investigate further:** Find the number of odd divisors of 360.

2. A positive integer x leaves a remainder of 3 when divided by 5, and a remainder of 5 when divided by 7. Find the smallest possible value of x.

Answer: 33

Method: List numbers that are 5 more than a multiple of 7: $5, 12, 19, 26, 33, \ldots$. Check these modulo 5: $33 \equiv 3 \pmod 5$, so 33 is the smallest such integer.

> **Investigate further:** Find the next smallest positive integer satisfying these conditions.

3. What is the sum of the digits of the number $2^{2024} \times 5^{2025}$?

Answer: 5

Method: Group powers of 10: $2^{2024} \times 5^{2025} = 2^{2024} \times 5^{2024} \times 5 = 10^{2024} \times 5 = 5 \times 10^{2024}$. This is a 5 followed by 2024 zeros, so the sum of the digits is 5.

> **Investigate further:** What is the sum of the digits of $4^{1012} \times 5^{2026}$?

4. Solve for the smallest positive integer x such that $4x \equiv 5 \pmod 7$.

Answer: 3

Method: Test small values of x: $4(1) = 4$, $4(2) = 8 \equiv 1$, $4(3) = 12 \equiv 5 \pmod 7$. The smallest positive integer is 3.

> **Investigate further:** Solve for x in $5x \equiv 4 \pmod 7$.

5. Find the last two digits of 99^2.

Answer: 01

Method: Use binomial expansion: $99^2 = (100 - 1)^2 = 10000 - 200 + 1 = 9801$. The last two digits are 01.

> **Investigate further:** Find the last three digits of 99^3.

6. Find the greatest common divisor of $2^{12} - 1$ and $2^{18} - 1$.

Answer: 63

Method: Use the property $\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1$. Here, $\gcd(12, 18) = 6$, so the answer is $2^6 - 1 = 64 - 1 = 63$.

> **Investigate further:** Find the greatest common divisor of $3^{12} - 1$ and $3^{18} - 1$.

7. The sum of three consecutive positive integers is a perfect cube. Find the smallest possible value for the largest of these three integers.

Answer: 10

Method: Let the integers be $n - 1, n, n + 1$. Their sum is $3n$. We need $3n = k^3$ for some integer k. The smallest valid k that is a multiple of 3 is 3, giving $3n = 27 \implies n = 9$. The largest integer is $n + 1 = 10$.

> **Investigate further:** What if the sum of five consecutive integers is a perfect cube?

8. Find the smallest positive integer n such that $10n$ is a perfect square and $6n$ is a perfect cube.

Answer: 36000

Method: Let $n = 2^x 3^y 5^z$. For $10n = 2^{x+1}3^y 5^{z+1}$ to be a square, $x+1$ is even, y is even, and $z+1$ is even. For $6n = 2^{x+1}3^{y+1}5^z$ to be a cube, $x+1, y+1, z$ are multiples of 3. We solve the modular congruences for x, y, z to get $x = 5, y = 2, z = 3$. Thus $n = 2^5 \cdot 3^2 \cdot 5^3 = 36000$.

> **Investigate further:** Find the smallest positive integer n such that $2n$ is a square and $3n$ is a cube.

9. Find the number of trailing zeros in 100!.

Answer: 24

Method: Count the number of factors of 5 in 100!. Using Legendre's formula: $\lfloor 100/5 \rfloor + \lfloor 100/25 \rfloor = 20 + 4 = 24$.

> **Investigate further:** Find the number of trailing zeros in 1000!.

10. The 5-digit number $24x8y$ is divisible by 4 and by 9. Find the maximum possible value of $x + y$.

Answer: 13

Method: Divisibility by 4 requires $8y$ to be divisible by 4, so $y \in \{0, 4, 8\}$. Divisibility by 9 requires the sum $14 + x + y$ to be a multiple of 9. Testing $y = 8 \implies 22 + x \equiv 0 \pmod 9 \implies x = 5$, yielding $x + y = 13$. Testing $y = 4 \implies x = 9$ or 0, yielding max $x + y = 13$. The maximum is 13.

> **Investigate further:** Find the minimum possible value of $x + y$ for the same constraints.

Section C Substitution & Structure

1. Find the sum of all integers x such that $\dfrac{x^2+5}{x+1}$ is an integer.

Answer: -8

Method: Rewrite as algebraic division: $\dfrac{x^2-1+6}{x+1} = x - 1 + \dfrac{6}{x+1}$. For this to be an integer, $x+1$ must divide 6. The divisors of 6 are $\pm1, \pm2, \pm3, \pm6$. We have $x + 1 = d \implies x = d - 1$. Summing $(d-1)$ over all 8 divisors gives $0 - 8(1) = -8$.

> **Investigate further:** Find the sum of all integers x such that $\frac{x^2+5}{x-1}$ is an integer.

2. Find the number of ordered pairs of positive integers (a, b) such that $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{6}$.

Answer: 9

Method: Clear denominators to get $ab - 6a - 6b = 0$. Apply Simon's Favourite Factoring Trick: add 36 to both sides to get $(a-6)(b-6) = 36$. The number of positive divisors of 36 is 9. Since $a, b > 0$, negative divisors do not yield valid positive integer solutions, leaving 9 pairs.

> **Investigate further:** Find the number of pairs if the RHS is $\frac{1}{10}$.

3. For how many integers $n \in [1, 100]$ is the polynomial $x^n - 1$ divisible by $x^2 + x + 1$?

Answer: 33

Method: The roots of $x^2 + x + 1 = 0$ are the non-real cube roots of unity, ω and ω^2. For $x^n - 1$ to be divisible by x^2+x+1, ω must be a root of $x^n-1=0$, meaning $\omega^n=1$. This occurs if and only if n is a multiple of 3. There are $\lfloor 100/3 \rfloor = 33$ such integers.

> **Investigate further:** For how many integers is x^n-1 divisible by x^2-x+1?

4. The 5-digit number $A34B2$, where A and B are single digits and $A \neq 0$, is a multiple of 99. Find the value of the product $A \times B$. *(after SMC 2018 Q16)*

Answer: 18

Method: A number is divisible by 99 if it is divisible by both 9 and 11. Divisibility by 9 means the sum of digits $A+3+4+B+2 = A+B+9$ is a multiple of 9, so $A+B$ is a multiple of 9. Since A and B are digits and $A \neq 0$, $A+B \in \{9, 18\}$. Divisibility by 11 means the alternating sum $A-3+4-B+2 = A-B+3$ is a multiple of 11, so $B-A \equiv 3 \pmod{11}$. Since $-9 \leq B-A \leq 9$, we have $B-A \in \{-8, 3\}$. Case 1: $B-A=3$. If $A+B=9$, $2B=12 \implies B=6, A=3$, yielding valid digits. If $A+B=18$, $2B=21$ (no solution). Case 2: $B-A=-8$. If $A+B=9$, $2B=1$ (no solution). If $A+B=18$, $2B=10 \implies B=5, A=13$ (invalid digit). Thus the only valid solution is $A=3, B=6$, so $A \times B = 3 \times 6 = 18$.

> **Investigate further:** The original SMC 2018 Q16 asks for a missing digit in a multiple of 11. This question builds on it by requiring simultaneous divisibility by 9 and 11, setting up a system of equations for the sum and difference of two digits. Can you find a 5-digit multiple of 99 in the form $A234B$?

5. The product of 24 integers is equal to 1. Their sum is 0. How many of the integers are equal to -1? *(after SMC 2015 Q11)*

Answer: 12

Method: Since the product of the integers is 1, each integer must be either 1 or -1. Let there be m ones and n negative ones. We are given the total number of integers $m+n=24$, and their sum $m-n=0$. Solving this simple system yields $m=12$ and $n=12$. We quickly verify that twelve -1s produce a product of $(-1)^{12}=1$, which satisfies all conditions. Thus, exactly 12 of the integers are -1.

> **Investigate further:** The original SMC 2015 Q11 asks a similar parity puzzle about 22 integers, which has no solution. This question modifies the total to 24 to guarantee a valid, elegant integer solution. What happens if the product of 24 integers is -1 and their sum is 0?

6. Find the largest integer x such that 2^x divides $3^{256}-1$.

Answer: 10

Method: Use difference of squares repeatedly: $3^{256}-1 = (3-1)(3+1)(3^2+1)(3^4+1)\dots(3^{128}+1)$. The term $3-1=2$ provides one factor of 2. The term $3+1=4$ provides two factors of 2. The remaining 7 terms are of the form $3^{2^k}+1 \equiv 1+1 \equiv 2 \pmod{4}$, meaning they each contribute exactly one factor of 2. Thus $x = 1+2+7(1) = 10$.

> **Investigate further:** Find the largest integer x such that 2^x divides $5^{128}-1$.

7. Find all pairs of primes (p, q) such that $p^2 - 2q^2 = 1$.

Answer: $(3, 2)$

Method: Rearrange as $p^2-1=2q^2$, so $(p-1)(p+1)=2q^2$. Since p is prime, it must be odd (if $p=2$, $-2q^2=3$ has no solution). Thus $p-1$ and $p+1$ are consecutive even numbers, so one is a multiple of 2 and the other a multiple of 4. If $q > 2$ is an odd prime, it cannot divide both $p-1$ and $p+1$. Thus $q = 2$, giving $p^2 - 8 = 1 \implies p = 3$.

> **Investigate further:** Find all pairs of primes (p, q) such that $p^2 - q^2 = 24$.

8. A palindromic 6-digit number N is formed by placing the digits of a 3-digit number x followed by the same digits in reverse order (for example, if $x = 123$, $N = 123321$). What is the largest prime number that must be a factor of all such numbers N? *(after TMUA 2019 Paper 1 Q12)*

Answer: 11

Method: Let x have digits a, b, c. Then $N = 100000a + 10000b + 1000c + 100c + 10b + a = 100001a + 10010b + 1100c$. Factoring out 11, we get $N = 11(9091a + 910b + 100c)$, so 11 always divides N. This matches the alternating sum rule: $(a + b + c) - (c + b + a) = 0$, which is trivially a multiple of 11. To see if any larger prime divides all such N, we can check $x = 100 \implies N = 100001 = 11 \times 9091$ and $x = 101 \implies N = 101101 = 11 \times 9191$. Their difference is $1100 = 11 \times 2^2 \times 5^2$. Neither 2 nor 5 divides 100001, so their greatest common divisor is exactly 11. Thus 11 is the largest prime that must be a factor.

> **Investigate further:** The original TMUA question explores the difference between a 3-digit number and its reverse. This question adapts the concept to a palindromic concatenation, creating a classic setup for the divisibility by 11 rule. What is the largest integer (not necessarily prime) that divides all such N?

Section D Challenge

1. Find the sum of all integers n such that $\dfrac{n^3 + 100}{n + 10}$ is an integer.

Answer: -540

Method: Use polynomial division to write $\dfrac{n^3 + 100}{n + 10} = n^2 - 10n + 100 - \dfrac{900}{n + 10}$. For this to be an integer, $n + 10$ must divide 900. Thus $n = d - 10$ for each integer divisor d of 900. The sum of $d - 10$ over all positive and negative divisors d leaves only the -10 terms, because $\sum d = 0$. Since $900 = 2^2 \times 3^2 \times 5^2$, it has $3 \times 3 \times 3 = 27$ positive divisors, and 54 total divisors. Sum is $-10 \times 54 = -540$.

> **Investigate further:** Find the sum of all integers n such that $\frac{n^2+100}{n+10}$ is an integer.

2. How many 5-digit numbers can be formed using only the digits 1, 2, and 3, such that the number is a multiple of 3?

Answer: 81

Method: A number is a multiple of 3 if the sum of its digits is a multiple of 3. We can choose the first four digits entirely freely, giving $3^4 = 81$ combinations. Let the sum of these four digits be S. Since the available last digits $\{1, 2, 3\}$ cover all three possible remainders modulo 3, there is exactly one valid choice for the final digit that makes $S + d \equiv 0 \pmod 3$. Thus, there are exactly 81 such numbers.

> **Investigate further:** How many 6-digit numbers can be formed using only the digits $1, 2, 3$ that are multiples of 3?

3. Let $P(x)$ be a polynomial with integer coefficients such that $P(1) = 10$ and $P(5) = 2$. Find all possible integer roots of $P(x)$.

Answer: 6

Method: If r is an integer root, then $P(r) = 0$. By polynomial divisibility, $(r-a)$ must divide $P(r)-P(a)$. Therefore, $(r-1)$ divides $P(r) - P(1) = -10$, so $r - 1 \in \{\pm1, \pm2, \pm5, \pm10\}$. Similarly, $(r - 5)$ divides $P(r)-P(5) = -2$, so $r-5 \in \{\pm1, \pm2\}$. Testing the overlap of these conditions yields $r \in \{3, 6\}$. However, $P(x) = (x - r)Q(x)$. If $r = 3$, $Q(1) = -5$ and $Q(5) = 1$. The difference $Q(5) - Q(1) = 6$ must be divisible by $5 - 1 = 4$, which is false. For $r = 6$, $Q(1) = -2$ and $Q(5) = -2$, which is perfectly valid (e.g., $Q(x) = -2$). Thus 6 is the only possible root.

> **Investigate further:** Let $P(x)$ have $P(2) = 7$ and $P(6) = 3$. Find all integer roots.

4. Find the number of positive divisors of the smallest positive integer n such that $\dfrac{n}{2}$ is a perfect square, $\dfrac{n}{3}$ is a perfect cube, and $\dfrac{n}{5}$ is a perfect fifth power.

Answer: 1232

Method: Let $n = 2^a 3^b 5^c$. $\frac{n}{2}$ is a square implies a is odd, and b, c are even. $\frac{n}{3}$ is a cube implies $b \equiv 1 \pmod 3$, and a, c are multiples of 3. $\frac{n}{5}$ is a fifth power implies $c \equiv 1 \pmod 5$, and a, b are multiples of 5. The smallest a satisfying $a \equiv 1 \pmod 2, a \equiv 0 \pmod{15}$ is 15. The smallest b satisfying $b \equiv 0 \pmod 2, b \equiv 0 \pmod 5, b \equiv 1 \pmod 3$ is 10. The smallest c satisfying $c \equiv 0 \pmod 6, c \equiv 1 \pmod 5$ is 6. Thus $n = 2^{15} 3^{10} 5^6$. The number of divisors is $(15 + 1)(10 + 1)(6 + 1) = 16 \times 11 \times 7 = 1232$.

> **Investigate further:** Find the prime factorization of the smallest such n.

5. Find all pairs of positive integers (x, y) such that $x^2 - y! = 2016$.

Answer: $(84, 7)$

Method: Rewrite as $x^2 = y! + 2016$. If $y \geq 9$, $y!$ is a multiple of 27 (since 9! contains $\lfloor 9/3 \rfloor + \lfloor 9/9 \rfloor = 4$ factors of 3). Thus $y! \equiv 0 \pmod{27}$. Also, $2016 = 27 \times 74 + 18 \equiv 18 \pmod{27}$. Thus $x^2 \equiv 18 \pmod{27}$. However, quadratic residues modulo 27 are $0, 1, 4, 7, 9, 10, 13, 16, 19, 22, 25$. Since 18 is not a quadratic residue, there are no solutions for $y \geq 9$. We manually check $y \leq 8$: for $y = 7$, $x^2 = 5040 + 2016 = 7056 \implies x = 84$. Checking all others up to 8 yields no other squares. Thus $(84, 7)$ is the only solution.

> **Investigate further:** Find all pairs (x, y) such that $x^2 - y! = 12$.