# Daily Logic Drill #7

*Sheet by David Akinyele-Aje*

| **Section** | **Topic** | **Questions** | **Target time** |
|---|---|---|---|
| A | Rapid Recognition | 10 | 2 min 30 sec |
| B | Manipulation Drills | 10 | 8 min |
| C | Substitution & Structure | 8 | 10 min |
| D | Challenge | 5 | 15 min |

*New toolkit today: Mixed Synthesis Capstone · Every technique from Days 1–6, combined.*

*Attempt each question before checking answers. Time yourself. No calculators.*

## Section A Rapid Recognition (≤15 s each)

*One quick question per technique from the whole week. Pure recognition, full speed.*

**1.** State the contrapositive of: "If a sequence is convergent, then it is bounded."

**2.** What is the negation of: "No integer is both even and odd"?

**3.** Is "$n$ is a multiple of 6" necessary, sufficient, both, or neither for "$n$ is a multiple of 2"?

**4.** To prove "there is no largest prime number" by contradiction, what should you assume first?

**5.** Give a counterexample to: "Every odd number is prime."

**6.** Name the fallacy: "If it is raining, the ground is wet. The ground is wet. Therefore it is raining."

**7.** What is the converse of "If $p$ then $q$"?

**8.** To prove directly that "if $n$ is a multiple of 7, then $n^2$ is a multiple of 49", what is the very first line of the proof?

**9.** True or false: "$\exists x\, \forall y\, (x < y)$ and $\forall y\, \exists x\, (x < y)$ are logically equivalent statements about real numbers."

**10.** Which proof technique is most natural for disproving "every multiple of 3 is odd"?

## Section B Manipulation Drills (≤50 s each)

*Each question pairs two techniques from different days.*

**1.** Prove by contrapositive: "If $n$ is not a multiple of 3 and not a multiple of 5, then $n$ is not a multiple of 15."

**2.** Prove by contradiction: "There is no positive integer $n$ such that $n^2 + n$ is odd."

**3.** Prove by induction: $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all positive integers $n$.

4. A student wants to prove "if $n$ is not a multiple of 5, then $n^2$ is not a multiple of 25" by instead showing "if $n$ is a multiple of 5, then $n^2$ is a multiple of 25." Is this a valid proof of the original claim? Justify your answer.

5. Give a counterexample showing that "$n$ is prime" is not sufficient for "$n$ is odd."

6. Negate the statement: "For every set $A$, there exists a set $B$ such that $A \cap B = \emptyset$." Is the negated statement true or false?

7. Prove by induction that $n^3 + 2n$ is divisible by 3 for all positive integers $n$.

8. Prove by contradiction that $\sqrt{6}$ is irrational.

9. For a polynomial $f$, classify "$f(x) > 0$ for all real $x$" as necessary, sufficient, both, or neither for "$f$ has no real roots."

10. A student proves $P(n)$ for all $n \geq 1$ by checking $P(1)$ directly, then showing $P(k) \implies P(k+1)$ only for $k \geq 2$ (skipping $k = 1$). Explain the gap this leaves in the proof.

## Section C Substitution & Structure (≤90 s each)

*Tricky synthesis MCQs, combining techniques from across the week.*

1. Let $P$: "$n$ is divisible by 4" and $Q$: "$n$ is divisible by 2 and $n$ is divisible by 6". Which of the following is true?

   A) $P \iff Q$.

   B) $P$ is sufficient but not necessary for $Q$.

   C) $P$ is necessary but not sufficient for $Q$.

   D) $P$ and $Q$ are neither necessary nor sufficient for each other — e.g. $n = 8$ satisfies $P$ but not $Q$, and $n = 6$ satisfies $Q$ but not $P$.

   E) $Q$ is impossible to satisfy.

2. A theorem states: "If every element of a finite set $S$ of positive integers is even, then the sum of all elements of $S$ is even." A student, wanting to prove the converse, argues: "Take $S = \{2, 4\}$; the sum is 6, which is even, and every element is indeed even, confirming the converse." What is the flaw?

   A) The theorem itself is false.

   B) A single supporting example cannot establish a universal converse statement — e.g. $S = \{1, 3\}$ has an even sum (4) despite containing no even elements at all, disproving the converse.

   C) $S = \{2, 4\}$ does not actually have an even sum.

   D) The original theorem's proof technique (not the converse) is what's flawed.

   E) There is no flaw; the converse is proved.

**3.** Which proof technique is most directly suited to establishing: "There exists a positive integer $n$ such that $n^2 + 3n + 1$ is not prime"?

A) Induction.

B) Direct exhibition of a specific such $n$ (e.g. $n = 6$: $6^2 + 3(6) + 1 = 55 = 5 \times 11$).

C) Contradiction.

D) Contrapositive.

E) Strong induction.

**4.** A student claims: "$n$ is a multiple of 9" is necessary and sufficient for "the digit sum of $n$ is a multiple of 9". They verify this only for $n = 9, 18, 27$ and declare the claim proved. Evaluate this.

A) The claim and the verification are both fully correct.

B) The claim is true (it is a standard divisibility rule), but checking three cases does not constitute a proof of a necessary-and-sufficient claim holding for every positive integer $n$.

C) The claim is false; digit sums have nothing to do with divisibility by 9.

D) The claim is true only for necessity, not sufficiency.

E) The claim is true only for sufficiency, not necessity.

**5.** What is the contrapositive of "If $n^2$ is not divisible by 4, then $n$ is odd"?

A) If $n$ is odd, then $n^2$ is not divisible by 4.

B) If $n$ is even, then $n^2$ is divisible by 4.

C) If $n^2$ is divisible by 4, then $n$ is even.

D) If $n$ is not odd, then $n^2$ is not divisible by 4.

E) If $n^2$ is divisible by 4, then $n$ is odd.

**6.** A student claims $P(n)$: "$n^2+5n+2$ is even for every positive integer $n$" is provable by induction: base case $P(1)$: $1 + 5 + 2 = 8$ is even; inductive step: assume $k^2 + 5k + 2$ is even; then $(k+1)^2 + 5(k+1) + 2 = k^2 + 7k + 8 = (k^2 + 5k + 2) + 2k + 6$, which is (even)+(even)=even, so $P(k+1)$ holds. Is this induction valid, and is the underlying claim true?

A) The induction is valid, and the claim is true for every positive integer — confirmed directly too, since $n^2 + 5n + 2 = n(n + 5) + 2$, and $n(n + 5)$ (a product of two integers of opposite parity, since 5 is odd) is always even.

B) The induction has a flaw in the algebra.

C) The claim is false — there exists some $n$ for which $n^2 + 5n + 2$ is odd.

D) The base case is wrong.

**7.** Consider positive integers $m, n$. Statement $P$: "For every prime $p$, if $p \mid mn$ then $p \mid m$ or $p \mid n$." Statement $Q$: "There exists a prime $p$ such that $p \mid mn$ but $p \nmid m$ and $p \nmid n$." Which of the following is true?

A) $P$ and $Q$ can both be true simultaneously.

B) $Q$ is the negation of $P$, and since $P$ is a true theorem (Euclid's Lemma) for all $m, n$, $Q$ is always false.

C) $P$ is the negation of $Q$.

D) Neither $P$ nor $Q$ is ever true.

E) $P$ is false in general.

**8.** A student claims: "$x = 1$" is necessary and sufficient for "$x^3 - 1 = 0$" (for real $x$), reasoning: "Sufficient, because substituting $x = 1$ gives 0. Necessary, because $1^3 - 1 = 0$, which confirms $x = 1$ works." Evaluate the necessity argument specifically.

A) Both parts of the reasoning are fully correct as given.

B) The "necessity" argument is circular: showing $1^3 - 1 = 0$ just re-verifies sufficiency again (that $x = 1$ satisfies the equation), and says nothing about whether $x = 1$ is the ONLY solution. Necessity IS true here — via $x^3 - 1 = (x - 1)(x^2 + x + 1)$, where the quadratic factor has no real roots (discriminant $1 - 4 = -3 < 0$) — just not established by the reasoning given.

C) The necessity argument is flawed because $x = -1$ is also a solution.

D) Sufficiency is actually false.

E) There is no correct way to establish necessity here.

## Section D Challenge (SMC / BMO1 difficulty)

*Full capstone proofs, drawing on every technique from the week.*

**1.** Long John Silverman has buried treasure at an integer point $(x, y)$ (not necessarily positive). He records the two values $x^2 + y$ and $x + y^2$; these two recorded numbers are distinct from each other. Prove that no other integer point $(x', y') \neq (x, y)$ can produce the same pair of recorded values (i.e. satisfy both $x'^2 + y' = x^2 + y$ and $x' + y'^2 = x + y^2$) — so the map pins down the treasure's location uniquely. *(after BMO1 2010 Q6)*

**2.** Two bags contain $m$ and $n$ balls respectively $(m, n > 0)$. Two operations are allowed, applied any number of times in any order: (a) remove the same number of balls from each bag; (b) triple the number of balls in one (chosen) bag. *(after BMO1 2012 Q4)*
Prove that if $m - n$ is odd, it is impossible to ever empty both bags (i.e. reach $(0, 0)$) using only these two operations.

**3.** Prove that there do not exist positive integers $a < b < c$ with $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = 1$ and $a, b, c$ all even.

**4.** Prove, using strong induction, that every positive integer $n \geq 2$ can be written as a product of primes in exactly one way, up to the order of the factors (the uniqueness part of the Fundamental Theorem of Arithmetic) — assuming as given Euclid's Lemma: if a prime $p$ divides a product $ab$, then $p \mid a$ or $p \mid b$.

**5.** Consider the statement $(*)$: "For all positive integers $m, n$: $m$ and $n$ are coprime if and only if there exist integers $x, y$ (not necessarily positive) with $mx + ny = 1$" (Bézout's Identity).
(a) Prove the $\Longleftarrow$ direction of $(*)$ directly.
(b) Taking the $\Longrightarrow$ direction of $(*)$ as given (it is normally proved via the Euclidean algorithm),

use $(*)$ to prove: if $p$ is prime and $p \nmid a$, then there exist integers $x, y$ with $px + ay = 1$; hence deduce Euclid's Lemma (if $p \mid ab$ then $p \mid a$ or $p \mid b$) as a consequence.

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*"Speed comes from recognition, not from rushing. Every shortcut is a pattern you have internalised."*

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Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths