Daily Logic Drill #6

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Mathematical Induction · Strong Induction · Base Case Traps · Misapplied Induction.

Attempt each question before checking answers. Time yourself. No calculators.

Section A Rapid Recognition ($\leq$15 s each)

Induction structure, base cases, and inductive steps. Pure recognition.

1. What are the two essential components required for a proof by mathematical induction?

2. To prove $P(n)$ for all integers $n \geq 4$ by induction, what is the base case?

3. If $P(k) \implies P(k+1)$ is proved for all $k \geq 1$, but no base case is ever established, has $P(n)$ been proved for any n?

4. In a proof of $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by induction, what statement exactly is assumed as the inductive hypothesis?

5. True or false: "Mathematical induction can be used to prove statements about every real number $x \geq 1$."

6. To prove $2^n > n^2$ for all $n \geq 5$ by induction, what is the smallest value of n for which the base case must be verified?

7. What is the negation of the inductive-step statement "For all $k \geq 1$, $P(k) \implies P(k+1)$"?

8. True or false: "If $P(1)$ is true and $P(k) \implies P(k+2)$ holds for all $k \geq 1$, then $P(n)$ is true for every positive integer n."

9. What is the smallest positive integer n for which $n! > 3^n$?

10. In a proof by strong induction, what may the inductive step assume that ordinary ("weak") induction's inductive step may not?

Section B Manipulation Drills ($\leq$50 s each)

Algebraic manipulations within inductive steps, and short full inductions.

1. Express $\sum_{i=1}^{k+1} i^3$ in terms of $\sum_{i=1}^{k} i^3$.

2. Assuming $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$, add $(k+1)$ and simplify to show $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

3. Prove by induction: $\sum_{i=1}^{n}(2i-1) = n^2$ for all positive integers n.

4. Assuming $4^{2k} - 1$ is divisible by 15, express $4^{2(k+1)} - 1$ in a form that shows it is also divisible by 15.

5. Prove by induction that $5^n - 1$ is divisible by 4 for all positive integers n.

6. Assuming $k! > 2^k$ for some $k \geq 4$, prove $(k+1)! > 2^{k+1}$.

7. Prove by induction: $2^n \geq n+1$ for all non-negative integers n.

8. Assuming $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$, show that $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$.

9. Prove by induction that the sum of the interior angles of a convex n-gon is $(n-2) \times 180°$ for all $n \geq 3$.

10. A student wants to prove $P(n)$ for all $n \geq 1$ using base cases $P(1)$ and $P(2)$, with inductive step "$P(k)$ and $P(k+1)$ together imply $P(k+2)$" for $k \geq 1$. Explain why establishing *both* $P(1)$ and $P(2)$ (not just $P(1)$) is necessary here.

Section C Substitution & Structure (≤90 s each)

Misapplied induction and subtle inductive-step gaps. Tricky MCQs.

1. A student attempts to prove $P(n)$: "in any group of n people, everyone has the same favourite integer," by induction. Base case: $P(1)$ is trivially true. Inductive step: assume $P(k)$ holds for any group of k people. Given a group of $k+1$ people, remove one person, leaving a group of k people who all share a favourite by $P(k)$; then put that person back and remove a different person, leaving another group of k people who also all share a favourite by $P(k)$. Since the two groups overlap, everyone must share the same favourite, so $P(k+1)$ holds. Which of the following identifies the flaw?

A) The base case $P(1)$ is invalid.

B) The inductive step secretly requires $k \geq 2$ for the two subgroups of size k to overlap; the very first application, from $P(1)$ to $P(2)$, has $k = 1$, where the two subgroups do not overlap at all, so the chain of implications never actually gets started.

C) The claim is actually true, and the proof is valid.

D) The flaw is that people cannot have favourite integers.

E) The inductive hypothesis is applied twice, which is never allowed.

2. A student attempts to prove: for all positive integers n, $n^3 - n$ is divisible by 6. Inductive step: "Assume $k^3 - k = 6m$ for some integer m. Since $(k+1)^3 - (k+1)$ is also divisible by 6, by the general theory of consecutive-integer products, $P(k+1)$ holds." What is the flaw?

A) $n^3 - n$ is not actually always divisible by 6.

B) The inductive step never actually derives $(k+1)^3-(k+1)$ from the assumption $k^3-k=6m$; it bypasses the inductive hypothesis entirely, appealing to an external fact instead of doing the algebra that connects $(k+1)^3-(k+1)$ back to k^3-k.

C) The base case is missing.

D) m must be shown to be positive.

E) There is no flaw.

3. A student wants to prove $P(n)$: "$n^2 > 3n+4$" for all sufficiently large n, and sets the base case at $n=4$, checking $16 > 16$. Since this is false, the student concludes the claim can never be proved by induction. What is the actual smallest valid base case, and what was wrong with the student's reasoning?

A) $n=3$; the student made an arithmetic slip.

B) $n=5$; $P(4)$ is false ($16 > 16$ fails), but $P(5)$ is true ($25 > 19$), and the inductive step $P(k) \implies P(k+1)$ can be verified to hold for $k \geq 5$ — a false $P(n_0)$ at one value doesn't mean no valid base case exists elsewhere.

C) $n=6$; the student needed to check two consecutive cases before starting.

D) There is no valid base case; the claim is false for all n.

E) $n=4$ actually does work; $16 > 16$ should be interpreted as $\geq$.

4. Every integer $n \geq 2$ can be written as a product of primes. A student attempts ordinary (single-step) induction: base case $n=2$ (prime); inductive step: "assume k is a product of primes; show $k+1$ is a product of primes." What is the fundamental problem with this approach?

A) There is no problem; the proof works fine.

B) If $k+1$ is composite, say $k+1=ab$ with $1 < a, b < k+1$, knowing that k (not a or b) is a product of primes gives no information about a or b at all — the inductive hypothesis about k is simply irrelevant to factoring $k+1$. Strong induction (assuming the claim for *all* of $2, \ldots, k$, not just k itself) is needed instead.

C) The base case should be $n=1$.

D) $k+1$ is always prime if k is a product of primes.

E) The inductive hypothesis needs to also assume $k-1$ is a product of primes.

5. A student is asked to determine the smallest positive integer n_0 such that $P(n)$: "$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} > \frac{1}{2}$" holds for all $n \geq n_0$. Checking $n=1$: the sum is just $\frac{1}{2}$, so $P(1)$ is FALSE ($\frac{1}{2} > \frac{1}{2}$ fails). The student concludes: "since $P(1)$ fails, no base case exists and induction cannot be used to establish $P(n)$ for any range of n." Is this conclusion correct?

A) Yes.

B) No — $P(1)$ failing only rules out $n_0 = 1$; the claim could still be true from some larger n_0 onward (and indeed it is: at $n=2$, the sum is $\frac{1}{3}+\frac{1}{4}=\frac{7}{12} > \frac{1}{2}$).

C) Yes, because a failed base case always means the inductive step will also fail.

D) No, because $\frac{1}{2} > \frac{1}{2}$ is actually a true statement.

E) The claim is undecidable by induction in principle.

6. A student attempts to prove "every amount of postage $n \geq 8$ pence can be made using only 3p and 5p stamps" by induction, using base cases $n = 8, 9, 10$ (each directly constructible: $8 = 3 + 5$, $9 = 3 + 3 + 3$, $10 = 5 + 5$). What must the inductive step actually use to correctly bridge from an earlier case to $P(k + 1)$ for $k \geq 10$?

A) $P(k + 1)$ can be built by taking the construction for $P(k - 2)$ (which holds since $k - 2 \geq 8$, already established by an earlier base case or earlier inductive step) and adding one more 3p stamp — this is strong induction, using a case three steps back, not just $P(k)$ itself.

B) $P(k + 1)$ follows directly from $P(k)$ by adding a 1p stamp.

C) No inductive step is needed once three base cases are given.

D) $P(k + 1)$ cannot be constructed from any earlier case; a completely separate direct construction is required for every n.

E) The base cases $8, 9, 10$ are unnecessary.

7. A student checks that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$ (where F_i is the i-th Fibonacci number, $F_1 = F_2 = 1$) for $n = 1, 2, 3, 4$ and concludes the identity holds for all positive integers n. Which of the following is the most accurate assessment?

A) This is a complete and valid proof.

B) Checking finitely many cases is not a proof; a full argument requires an inductive step showing $F_1 + \cdots + F_{2k-1} = F_{2k} \implies F_1 + \cdots + F_{2k-1} + F_{2k+1} = F_{2k+2}$, which follows directly from the Fibonacci recurrence $F_{2k+2} = F_{2k+1} + F_{2k}$.

C) The identity is actually false, since Fibonacci numbers grow exponentially.

D) No proof is possible since Fibonacci numbers are not well understood.

E) Checking $n = 1, \ldots, 4$ is sufficient because Fibonacci identities always stabilise after four terms.

8. Which of the following correctly and precisely states what must be proved in the inductive step of a proof that $P(n)$ holds for all integers $n \geq 1$?

A) $P(n)$ is true for some particular large n.

B) $P(k)$ is true for all k.

C) For every integer $k \geq 1$: if $P(k)$ is true, then $P(k + 1)$ is true.

D) There exists an integer $k \geq 1$ such that $P(k) \implies P(k + 1)$.

E) $P(1) \implies P(n)$ for all n.

Section D Challenge

(SMC / BMO1 difficulty)

Full olympiad-style induction proofs.

1. Let x be a real number such that $t = x + \dfrac{1}{x}$ is an integer greater than 2. Define $t_n = x^n + \dfrac{1}{x^n}$ for each positive integer n (and $t_0 = 2$). *(after BMO1 2015 Q4)*
(a) Show that $t_{n+1} = t \cdot t_n - t_{n-1}$ for all $n \geq 1$.
(b) Prove by strong induction that t_n is an integer for every positive integer n.

2. A positive integer is called *charming* if it equals 2 or is of the form 3^i5^j for non-negative integers i, j. Prove, using strong induction, that every positive integer can be written as a sum of distinct charming integers. *(after BMO1 2016 Q6)*

3. Prove by induction that for all real $x > -1$ and all positive integers n, $(1+x)^n \geq 1 + nx$ (Bernoulli's Inequality). Identify precisely which step of your inductive argument requires the condition $x > -1$.

4. Let $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$ for $n \geq 1$. Prove by induction that $a_1, a_2, \ldots, a_n$ are pairwise coprime for every positive integer n (i.e. $\gcd(a_i, a_j) = 1$ whenever $i \neq j$).

5. Prove by induction that for all integers $n \geq 2$, $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$. (You may find it helpful to bound $\frac{1}{(n+1)^2}$ against $\frac{1}{n} - \frac{1}{n+1}$ in your inductive step.)

"Speed comes from recognition, not from rushing. Every shortcut is a pattern you have internalised."

Found a mistake or want to contribute a sheet?
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