Daily Logic Drill #7 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Mixed Synthesis Capstone · Every technique from Days 1–6, combined.

Section A

1. State the contrapositive of: "If a sequence is convergent, then it is bounded."

Answer: If a sequence is not bounded, then it is not convergent.

Method: The contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$: negate "bounded" to get "not bounded", and negate "convergent" to get "not convergent", then swap their order.

> **Investigate further:** Is the converse of the original statement ("if bounded, then convergent") actually true?

2. What is the negation of: "No integer is both even and odd"?

Answer: There exists an integer that is both even and odd.

Method: "No x is $P(x)$" means $\forall x\, \neg P(x)$; its negation is $\exists x\, P(x)$.

> **Investigate further:** Is the negated statement true or false?

3. Is "n is a multiple of 6" necessary, sufficient, both, or neither for "n is a multiple of 2"?

Answer: Sufficient but not necessary.

Method: Multiple of 6 $\implies$ multiple of 2 (since $6 = 2 \times 3$), so it's sufficient. But $n = 4$ is a multiple of 2 without being a multiple of 6, so it's not necessary.

> **Investigate further:** What condition on n would be both necessary and sufficient for "n is a multiple of 2"?

4. To prove "there is no largest prime number" by contradiction, what should you assume first?

Answer: Assume there exists a largest prime number (call it P).

Method: Proof by contradiction begins by assuming the negation of the statement to be proved.

> **Investigate further:** How would you then derive a contradiction from this assumption?

5. Give a counterexample to: "Every odd number is prime."

Answer: 9 (or $15, 21, \ldots$)

Method: 9 is odd, but $9 = 3 \times 3$ is composite, not prime.

> **Investigate further:** What is the smallest odd composite number?

6. Name the fallacy: "If it is raining, the ground is wet. The ground is wet. Therefore it is raining."

Answer: Affirming the consequent

Method: The form $P \implies Q$, Q, therefore P is invalid; the ground could be wet for other reasons (e.g. a sprinkler).

> **Investigate further:** What valid rule has the similar-looking but correct form $P \implies Q$, P, therefore Q?

7. What is the converse of "If p then q"?

Answer: "If q then p"

Method: The converse of $P \implies Q$ swaps the two sides to give $Q \implies P$ — logically a different (generally non-equivalent) statement.

> **Investigate further:** Give an example of a true implication whose converse is false.

8. To prove directly that "if n is a multiple of 7, then n^2 is a multiple of 49", what is the very first line of the proof?

Answer: Suppose n is a multiple of 7, so $n = 7k$ for some integer k.

Method: A direct proof of $P \implies Q$ begins by assuming P and introducing notation for it, here the definition of "multiple of 7".

> **Investigate further:** Complete the rest of this proof in one further line.

9. True or false: "$\exists x \, \forall y \, (x < y)$ and $\forall y \, \exists x \, (x < y)$ are logically equivalent statements about real numbers."

Answer: False

Method: $\exists x \, \forall y \, (x < y)$ claims a single x smaller than every real y — false (no smallest real number). $\forall y \, \exists x \, (x < y)$ claims every y has some smaller x — true (take $x = y - 1$). Different truth values, so not equivalent.

> **Investigate further:** In general, does $\exists x \, \forall y \, P(x, y)$ imply $\forall y \, \exists x \, P(x, y)$, or only the reverse?

10. Which proof technique is most natural for disproving "every multiple of 3 is odd"?

Answer: Counterexample

Method: A single value ($n = 6$, a multiple of 3 that is even) disproves the universal claim directly — no need for contradiction, induction, or contrapositive.

> **Investigate further:** Why would proof by contradiction be needlessly roundabout here?

Section B

1. Prove by contrapositive: "If n is not a multiple of 3 and not a multiple of 5, then n is not a multiple of 15."

Answer: Proof by contrapositive

Method: Contrapositive: "If n is a multiple of 15, then n is a multiple of 3 or a multiple of 5." Suppose $n = 15k$ for some integer k. Then $n = 3(5k)$, so n is a multiple of 3 (and, by the same token, $n = 5(3k)$ shows it is also a multiple of 5). Since the contrapositive is proved, the original statement holds.

> **Investigate further:** Is the contrapositive here actually stronger than what was needed — could you prove just "multiple of 3" alone?

2. Prove by contradiction: "There is no positive integer n such that $n^2 + n$ is odd."

Answer: Proof by contradiction

Method: Suppose toward a contradiction that $n^2 + n$ is odd for some positive integer n. But $n^2 + n = n(n + 1)$, a product of two consecutive integers, one of which is always even; hence $n(n + 1)$ is always even. This contradicts $n^2 + n$ being odd. So no such n exists.

> **Investigate further:** Could this also be proved directly, by splitting into cases on the parity of n?

3. Prove by induction: $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all positive integers n.

Answer: Proof by induction

Method: Base case $n = 1$: LHS$= 1$, RHS$= \frac{1\cdot 2\cdot 3}{6} = 1$. Holds.
Inductive step: assume $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. Then $\sum_{i=1}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)[k(2k+1)+6(k+1)]}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$, which matches the formula at $n = k + 1$. By induction, true for all positive integers n.

> **Investigate further:** Factor $2k^2 + 7k + 6$ to confirm it equals $(k + 2)(2k + 3)$.

4. A student wants to prove "if n is not a multiple of 5, then n^2 is not a multiple of 25" by instead showing "if n is a multiple of 5, then n^2 is a multiple of 25." Is this a valid proof of the original claim? Justify your answer.

Answer: No.

Method: The original claim is $\neg A \implies \neg B$ (where A: "$5 \mid n$", B: "$25 \mid n^2$"). Its valid contrapositive is $B \implies A$. The student instead proved $A \implies B$, a different (though also true) statement, which is neither the original claim nor its contrapositive.

> **Investigate further:** State the actual contrapositive of the original claim.

5. Give a counterexample showing that "n is prime" is not sufficient for "n is odd."

Answer: $n = 2$

Method: 2 is prime, but 2 is even, not odd — so "prime" does not guarantee "odd".

> **Investigate further:** Is "n is prime" necessary for "n is odd"?

6. Negate the statement: "For every set A, there exists a set B such that $A \cap B = \emptyset$." Is the negated statement true or false?

Answer: Negation: "There exists a set A such that for every set B, $A \cap B \neq \emptyset$." This negation is false.

Method: The original statement is true (for any set A, take $B = \emptyset$; then $A \cap \emptyset = \emptyset$ always), so its negation must be false. Indeed, for any candidate A, choosing $B = \emptyset$ gives $A \cap B = \emptyset$, contradicting the negation's claim that every B gives a nonempty intersection.

> **Investigate further:** Does the negation change if B is required to be non-empty?

7. Prove by induction that $n^3 + 2n$ is divisible by 3 for all positive integers n.

Answer: Proof by induction

Method: Base case $n = 1$: $1 + 2 = 3$, divisible by 3.
Inductive step: assume $k^3 + 2k = 3m$ for some integer m. Then $(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3k^2 + 3k + 3 = 3m + 3(k^2 + k + 1) = 3(m + k^2 + k + 1)$, divisible by 3. By induction, true for all positive integers n.

> **Investigate further:** Verify the claim directly for $n = 1, 2, 3, 4$.

8. Prove by contradiction that $\sqrt{6}$ is irrational.

Answer: Proof by contradiction

Method: Suppose toward a contradiction that $\sqrt{6} = p/q$ for coprime positive integers p, q. Then $6q^2 = p^2$. Since $2 \mid 6q^2$, $2 \mid p^2$, so $2 \mid p$ (as 2 is prime); write $p = 2r$. Then $6q^2 = 4r^2$, so $3q^2 = 2r^2$. Since $2 \mid 3q^2$ and $\gcd(2, 3) = 1$, $2 \mid q^2$, so $2 \mid q$. But then 2 divides both p and q, contradicting $\gcd(p, q) = 1$. Hence $\sqrt{6}$ is irrational.

> **Investigate further:** Where exactly does this proof use that p, q were chosen coprime?

9. For a polynomial f, classify "$f(x) > 0$ for all real x" as necessary, sufficient, both, or neither for "f has no real roots."

Answer: Sufficient but not necessary.

Method: If $f(x) > 0$ for all x, then certainly $f(x) \neq 0$ ever, so f has no real roots — sufficient. But $f(x) = -(x^2 + 1)$ has no real roots (it's always negative) without ever being positive — so not necessary.

> **Investigate further:** What condition on f's sign would be both necessary and sufficient for "no real roots"?

10. A student proves $P(n)$ for all $n \geq 1$ by checking $P(1)$ directly, then showing $P(k) \implies P(k+1)$ only for $k \geq 2$ (skipping $k = 1$). Explain the gap this leaves in the proof.

Answer: $P(2)$ is never derived, so the chain of implications never actually starts.

Method: The inductive step, starting at $k = 2$, would derive $P(3)$ from $P(2)$, $P(4)$ from $P(3)$, and so on — but it requires $P(2)$ to already be established, which was never shown (only $P(1)$ was checked directly, and the step from $P(1)$ to $P(2)$, i.e. $k = 1$, was explicitly skipped). So only $P(1)$ is actually proved; nothing beyond it follows.

> **Investigate further:** What is the minimal fix: include $k = 1$ in the inductive step, or add $P(2)$ as a second base case?

Section C

1. P: "n divisible by 4"; Q: "n divisible by 2 and by 6".

Answer: D

Method: $n = 8$: divisible by 4 (P true), but not by 6 (Q false) — so $P \not\Longleftrightarrow Q$. $n = 6$: divisible by 2 and 6 (Q true), but not by 4 (P false) — so $Q \not\Longleftrightarrow P$. Since 4 and 6 are not coprime (unlike $2, 3$), the usual "divisible by both factors $\Longleftrightarrow$ divisible by the product" shortcut does not apply.

> **Investigate further:** For which pair of numbers a, b (in place of 4 and the pair $2, 6$) would the analogous $P \Longleftrightarrow Q$ actually hold?

2. A student "proves" the converse of "every element even $\Longrightarrow$ sum even" using only $S = \{2, 4\}$.

Answer: B

Method: One supporting example never proves a universal converse. $S = \{1, 3\}$: both elements odd, yet the sum 4 is even — disproving the converse directly.

> **Investigate further:** Is the converse ever true for sets of size exactly 1?

3. Which technique suits "there exists n with $n^2 + 3n + 1$ not prime"?

Answer: B

Method: An existential claim is established by directly exhibiting one witness: $n = 6$ gives $6^2+3(6)+1 = 36 + 18 + 1 = 55 = 5 \times 11$, composite. Contradiction, contrapositive, and induction are all suited to universal or conditional claims, not existence claims.

> **Investigate further:** Would induction ever be an appropriate technique for an existence claim like this one?

4. "Multiple of 9 $\Longleftrightarrow$ digit sum multiple of 9", verified only for $n = 9, 18, 27$.

Answer: B

Method: The divisibility rule itself is a true, well-known theorem (provable via $10 \equiv 1 \pmod 9$, so a number's value mod 9 equals its digit sum mod 9). But three verified cases give no logical guarantee it holds for every positive integer n — a genuine proof needs the general modular argument, not finite checking.

> **Investigate further:** Sketch the general proof using $10 \equiv 1 \pmod 9$.

5. Contrapositive of "If n^2 is not divisible by 4, then n is odd".

Answer: B

Method: Original: $\neg A \Longrightarrow B$ where A: "$4 \mid n^2$", B: "n odd". Contrapositive: $\neg B \Longrightarrow A$, i.e. "if n is not odd (even), then $4 \mid n^2$" — matching option B exactly (verify: $n = 2k \Longrightarrow n^2 = 4k^2$, divisible by 4).

> **Investigate further:** Why is option C ("if n^2 divisible by 4 then n even") the converse rather than the contrapositive?

6. Induction proof that $n^2 + 5n + 2$ is always even.

Answer: A

Method: The induction shown is fully valid (the algebra $(k^2 + 5k + 2) + 2k + 6 = k^2 + 7k + 8$ checks out exactly). It's also confirmable directly: $n^2 + 5n + 2 = n(n + 5) + 2$, and $n(n + 5)$ is always even

(since n and $n+5$ have opposite parity, one of any pair differing by an odd number 5 must be even), so $n(n+5)+2$ is always even, for every positive integer n.

> **Investigate further:** Does this direct argument make the induction proof redundant, or do they serve different pedagogical purposes?

7. P: Euclid's Lemma (as a universal statement); Q: existence of a counterexample prime.

Answer: B

Method: Q is exactly $\neg P$ (a counterexample prime exists $\iff$ the universal claim is false). Euclid's Lemma is a genuine, well-known true theorem for all m, n (provable via unique prime factorization or Bézout's identity), so P is true and therefore $Q = \neg P$ is always false.

> **Investigate further:** How does Bézout's identity give a short proof of Euclid's Lemma? (See Section D, Q5.)

8. Necessity argument for "$x = 1 \iff x^3 - 1 = 0$".

Answer: B

Method: The student's "necessity" reasoning ($1^3 - 1 = 0$) merely re-checks that $x = 1$ satisfies the equation — exactly the same computation as the sufficiency check, and it says nothing about whether any OTHER real x might also satisfy $x^3 - 1 = 0$. This is circular: necessity requires ruling out other solutions, not re-confirming this one. Necessity IS actually true here: $x^3 - 1 = (x-1)(x^2+x+1)$, and the quadratic factor has discriminant $1 - 4 = -3 < 0$ (no real roots), so $x = 1$ genuinely is the only real solution — just not established by the reasoning given.

> **Investigate further:** What are the other two (complex) cube roots of unity?

Section D

1. Prove the treasure location (x, y) is uniquely determined by the pair $(x^2 + y,\ x + y^2)$, given the two values are distinct. *(after BMO1 2010 Q6)*

Answer: Proof below.

Method: Suppose $(x_1, y_1) \neq (x_2, y_2)$ both give the same recorded pair: $x_1^2 + y_1 = x_2^2 + y_2$ and $x_1 + y_1^2 = x_2 + y_2^2$. Let $d = x_1 - x_2$, $e = y_1 - y_2$.
From the first equation: $x_1^2 - x_2^2 = y_2 - y_1 \implies d(x_1 + x_2) = -e$.
From the second: $y_1^2 - y_2^2 = x_2 - x_1 \implies e(y_1 + y_2) = -d$.
Substituting the first into the second: $-d(x_1 + x_2)(y_1 + y_2) = -d \implies d\big[(x_1 + x_2)(y_1 + y_2) - 1\big] = 0$.
So either $d = 0$, or $(x_1 + x_2)(y_1 + y_2) = 1$.
Case $d = 0$: then $e = -d(x_1 + x_2) = 0$ too, so $(x_1, y_1) = (x_2, y_2)$ — not a distinct second point, as required.
Case $(x_1 + x_2)(y_1 + y_2) = 1$: since both factors are integers, either both equal 1, or both equal -1.
If $x_1 + x_2 = 1 = y_1 + y_2$: substituting $x_2 = 1 - x_1, y_2 = 1 - y_1$ into $e = -d(x_1 + x_2) = -d$ (i.e. $y_1 - y_2 = x_2 - x_1$) gives, after simplification, $x_1 + y_1 = 1$. But then $x_1^2 + y_1 = x_1^2 + (1 - x_1) = x_1^2 - x_1 + 1$ and $x_1 + y_1^2 = x_1 + (1 - x_1)^2 = x_1^2 - x_1 + 1$ — the two recorded values are EQUAL, contradicting the given hypothesis that they are distinct.
If $x_1 + x_2 = -1 = y_1 + y_2$: an identical computation (replacing 1 with -1 throughout) forces $y_1 = x_1$, and then $x_1^2 + y_1 = x_1^2 + x_1 = x_1 + y_1^2$ again — the two recorded values are equal, the same contradiction.
Both sub-cases contradict the hypothesis, so only $d = 0$ survives, proving $(x_1, y_1) = (x_2, y_2)$: the treasure's location is uniquely determined.

> **Investigate further:** Check directly that $(x_1, y_1) = (2, -1)$ and $(x_2, y_2) = (-1, 2)$ (which satisfy $x_1 + x_2 = 1 = y_1 + y_2$ and $x_1 + y_1 = 1$) both give the recorded pair $(3, 3)$, confirming why the "distinct values" hypothesis is essential — without it, this genuine second location would make the map ambiguous.

2. Bags with m, n balls; operations: remove equal amounts, or triple one bag. Prove $m - n$ odd $\implies$ can never reach $(0, 0)$. *(after BMO1 2012 Q4)*

Answer: Proof below, via the invariant $m - n$ mod 2.

Method: *Claim*: the value of $m - n \pmod 2$ is unchanged by either operation.
Removing k from both bags: new difference is $(m - k) - (n - k) = m - n$, exactly unchanged.
Tripling one bag, say $m \to 3m$: new difference is $3m - n$, and $(3m - n) - (m - n) = 2m$, which is even — so $3m - n \equiv m - n \pmod 2$, unchanged.
Since both operations preserve $m - n$ mod 2, this quantity is invariant throughout any sequence of operations.
The target state $(0, 0)$ has $m - n = 0$, which is even. So if the starting $m - n$ is odd, the invariant can never reach the even value required for $(0, 0)$ — meaning $(0, 0)$ is unreachable.

> **Investigate further:** Verify concretely for $m = 1, n = 2$ that a few rounds of operations (e.g. triple the 1, then remove 1 from both) always leave $m - n$ odd.

3. Prove no positive integers $a < b < c$, all even, satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$.

Answer: Proof below.

Method: Suppose $a < b < c$ are positive integers with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ (temporarily ignoring the "even" condition). Since $a < b < c$, $\frac{1}{a} > \frac{1}{b} > \frac{1}{c}$, so $1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{3}{a}$, giving $a < 3$, i.e. $a \in \{1, 2\}$.
If $a = 1$: $\frac{1}{b} + \frac{1}{c} = 0$, impossible for positive b, c.
So $a = 2$: $\frac{1}{b} + \frac{1}{c} = \frac{1}{2}$, with $b > 2$. Since $b < c$, $\frac{1}{b} > \frac{1}{c}$, so $\frac{1}{2} = \frac{1}{b} + \frac{1}{c} < \frac{2}{b}$, giving $b < 4$; combined with $b > 2$, $b = 3$. Then $\frac{1}{c} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, so $c = 6$.
So the unique solution to $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ with $a < b < c$ positive integers is $(a, b, c) = (2, 3, 6)$. Since $b = 3$ is odd, there is no solution with a, b, c all even.

> **Investigate further:** Why does the bounding argument $1 < \frac{3}{a}$ (rather than checking cases exhaustively) make this proof efficient?

4. Prove the uniqueness part of the Fundamental Theorem of Arithmetic, given Euclid's Lemma.

Answer: Proof by strong induction (via minimal counterexample) below.

Method: Suppose, for contradiction, that some integer $n \geq 2$ has two genuinely different prime factorisations. By well-ordering, let n be the SMALLEST such integer. Write $n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s$ as two different factorisations into primes (with multiplicity, not necessarily ordered the same way).
Since $p_1 \mid n = q_1 q_2 \cdots q_s$, repeated application of Euclid's Lemma gives $p_1 \mid q_i$ for some i; relabel so $p_1 \mid q_1$. Since q_1 is prime and $p_1 > 1$, this forces $p_1 = q_1$.
Then $n/p_1 = p_2 \cdots p_r = q_2 \cdots q_s$ is a positive integer strictly smaller than n (since $p_1 \geq 2$), and its two factorisations must still be genuinely different (otherwise, re-inserting $p_1 = q_1$ into matching factorisations would make the original two factorisations of n the same after all, contradicting our assumption that they differ).
So $n/p_1 < n$ is a smaller integer with two different factorisations — contradicting that n was the SMALLEST such counterexample. Hence no counterexample exists: every integer $n \geq 2$ has a unique prime factorisation, up to ordering.

> **Investigate further:** Where exactly does this proof use the "smallest counterexample" idea rather than a direct forward induction from $n = 2$ upward?

5. (a) Prove $mx + ny = 1 \implies \gcd(m, n) = 1$. (b) Deduce Euclid's Lemma from Bézout's Identity.

Answer: Proofs below.

Method: (a) Suppose $mx + ny = 1$ for some integers x, y. Let d be any common divisor of m and n. Since $d \mid m$ and $d \mid n$, d divides any integer combination $mx + ny$, in particular $d \mid 1$. So $d = 1$ (taking d positive), meaning the only common divisor is 1, i.e. $\gcd(m, n) = 1$: m, n are coprime.
(b) Suppose p is prime and $p \nmid a$. Since p is prime, its only positive divisors are 1 and p; as $p \nmid a$, $\gcd(p, a) \neq p$, so $\gcd(p, a) = 1$, i.e. p, a are coprime. By the $\implies$ direction of $(*)$ (given), there exist integers x, y with $px + ay = 1$.
Now suppose $p \mid ab$. Multiply the Bézout identity by b: $pbx + aby = b$. Since $p \mid pbx$ trivially, and $p \mid ab \implies p \mid aby$, the sum $pbx + aby = b$ is also divisible by p. Hence $p \mid b$.
So: if $p \mid ab$ and $p \nmid a$, then $p \mid b$. Equivalently, if $p \mid ab$, then $p \mid a$ or $p \mid b$ — Euclid's Lemma.

> **Investigate further:** Where in part (b) is the primality of p used, as opposed to just any integer p?