Daily Logic Drill #6 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Mathematical Induction · Strong Induction · Base Case Traps · Misapplied Induction.

Section A

1. What are the two essential components required for a proof by mathematical induction?

Answer: A base case, and an inductive step.

Method: The base case establishes $P(n_0)$ for the starting value; the inductive step proves $P(k) \implies P(k+1)$ for all relevant k. Together these imply $P(n)$ for every $n \geq n_0$.

> **Investigate further:** Why is neither component alone sufficient?

2. To prove $P(n)$ for all integers $n \geq 4$ by induction, what is the base case?

Answer: $P(4)$

Method: The base case is always the smallest value in the claimed range.

> **Investigate further:** If the claim were instead "for all $n \geq 4$ except $n = 5$", would ordinary induction still apply directly?

3. If $P(k) \implies P(k+1)$ is proved for all $k \geq 1$, but no base case is ever established, has $P(n)$ been proved for any n?

Answer: No.

Method: The inductive step alone is vacuous without a starting point — it is entirely possible for $P(k) \implies P(k+1)$ to hold for every k while $P(n)$ is false for every n (e.g. if $P(n)$ is always false, $P(k) \implies P(k+1)$ holds vacuously).

> **Investigate further:** Give an example of a statement $P(n)$ that is false for every n but for which $P(k) \implies P(k+1)$ holds for all k.

4. In a proof of $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by induction, what statement exactly is assumed as the inductive hypothesis?

Answer: $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$, for some fixed (but arbitrary) k.

Method: The inductive hypothesis assumes the formula holds at $n = k$, in order to derive it at $n = k+1$.

> **Investigate further:** Why must k be treated as arbitrary rather than a specific number?

5. True or false: "Mathematical induction can be used to prove statements about every real number

$x \geq 1$."

Answer: False

Method: Induction proves statements indexed by the integers (or a well-ordered discrete set), stepping from k to $k+1$. The real numbers between consecutive integers are never reached by this stepping process.

> **Investigate further:** What technique is typically used instead for statements about all real x in an interval?

6. To prove $2^n > n^2$ for all $n \geq 5$ by induction, what is the smallest value of n for which the base case must be verified?

Answer: $n = 5$

Method: Check: $2^5 = 32 > 25 = 5^2$, true. (Note $2^4 = 16 = 4^2$ and $2^3 = 8 < 9 = 3^2$, so the claim genuinely fails before $n = 5$ — this is why the range starts at 5.)

> **Investigate further:** Verify that the claim fails at $n = 4$ and $n = 3$.

7. What is the negation of the inductive-step statement "For all $k \geq 1$, $P(k) \implies P(k+1)$"?

Answer: There exists $k \geq 1$ such that $P(k)$ is true and $P(k+1)$ is false.

Method: $\neg(\forall k\,(P(k) \implies P(k+1))) \equiv \exists k\,\neg(P(k) \implies P(k+1)) \equiv \exists k\,(P(k) \wedge \neg P(k+1))$.

> **Investigate further:** Which earlier drill covered exactly this negation rule for implications?

8. True or false: "If $P(1)$ is true and $P(k) \implies P(k+2)$ holds for all $k \geq 1$, then $P(n)$ is true for every positive integer n."

Answer: False

Method: $P(1) \implies P(3) \implies P(5) \implies \cdots$ only establishes $P(n)$ for odd n. $P(2)$ is never reached, since nothing in the chain starts from an even index.

> **Investigate further:** What additional single fact would need to be assumed to fix this and cover every positive integer?

9. What is the smallest positive integer n for which $n! > 3^n$?

Answer: $n = 7$

Method: $6! = 720$ vs $3^6 = 729$: $720 < 729$, fails. $7! = 5040$ vs $3^7 = 2187$: $5040 > 2187$, holds. So $n = 7$ is the smallest.

> **Investigate further:** Verify $n! \leq 3^n$ for $n = 1, \ldots, 6$ explicitly.

10. In a proof by strong induction, what may the inductive step assume that ordinary ("weak") induction's inductive step may not?

Answer: The truth of $P(1), P(2), \ldots, P(k)$ (all previous cases), not just $P(k)$ alone.

Method: Weak induction's inductive step assumes only $P(k)$ to derive $P(k+1)$; strong induction's inductive step may use any or all of $P(1), \ldots, P(k)$.

> **Investigate further:** Are weak and strong induction actually logically equivalent in what they can prove, despite the difference in what's assumed?

Section B

1. Express $\sum_{i=1}^{k+1} i^3$ in terms of $\sum_{i=1}^{k} i^3$.

Answer: $\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$

Method: The sum up to $k+1$ terms equals the sum up to k terms plus the new $(k+1)$-th term.

> **Investigate further:** If $\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2$, verify the inductive step closes correctly.

2. Assuming $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$, add $(k+1)$ and simplify to show $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

Answer: $\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

Method: Factor out $(k+1)$ from both terms in the numerator.

> **Investigate further:** What is the base case for this induction, and does it check out?

3. Prove by induction: $\sum_{i=1}^{n}(2i-1) = n^2$ for all positive integers n.

Answer: Proof by induction

Method: Base case $n=1$: LHS $=1$, RHS $=1^2=1$. Holds.
Inductive step: assume $\sum_{i=1}^{k}(2i-1) = k^2$. Then $\sum_{i=1}^{k+1}(2i-1) = k^2+(2(k+1)-1) = k^2+2k+1 = (k+1)^2$.
So the formula holds for $k+1$. By induction, true for all positive integers n.

> **Investigate further:** What is the geometric interpretation of this identity (sum of first n odd numbers)?

4. Assuming $4^{2k}-1$ is divisible by 15, express $4^{2(k+1)}-1$ in a form that shows it is also divisible by 15.

Answer: $4^{2(k+1)} - 1 = 16(4^{2k}-1) + 15$

Method: $4^{2(k+1)} - 1 = 16 \cdot 4^{2k} - 1 = 16(4^{2k}-1) + 16 - 1 = 16(4^{2k}-1) + 15$. Since $4^{2k}-1$ is divisible by 15 (hypothesis) and 15 is divisible by 15, the sum is divisible by 15.

> **Investigate further:** Why does $4^{2k} = 16^k \equiv 1 \pmod{15}$ make this pattern unsurprising?

5. Prove by induction that $5^n - 1$ is divisible by 4 for all positive integers n.

Answer: Proof by induction

Method: Base case $n=1$: $5^1-1=4$, divisible by 4.
Inductive step: assume $5^k - 1 = 4m$ for some integer m. Then $5^{k+1} - 1 = 5 \cdot 5^k - 1 = 5(4m+1) - 1 = 20m + 4 = 4(5m+1)$, divisible by 4. By induction, true for all positive integers n.

> **Investigate further:** Generalise: for which integer a is $a^n - 1$ divisible by $a-1$ for every positive integer n?

6. Assuming $k! > 2^k$ for some $k \geq 4$, prove $(k+1)! > 2^{k+1}$.

Answer: Proof

Method: $(k+1)! = (k+1) \cdot k! > (k+1) \cdot 2^k$ (using the hypothesis). Since $k \geq 4$, $k+1 \geq 5 > 2$, so $(k+1) \cdot 2^k > 2 \cdot 2^k = 2^{k+1}$. Combining, $(k+1)! > 2^{k+1}$.

> **Investigate further:** Check the base case $k = 4$: is $4! > 2^4$?

7. Prove by induction: $2^n \geq n+1$ for all non-negative integers n.

Answer: Proof by induction

Method: Base case $n = 0$: $2^0 = 1 \geq 0 + 1 = 1$. Holds (with equality).
Inductive step: assume $2^k \geq k+1$. Then $2^{k+1} = 2 \cdot 2^k \geq 2(k+1) = 2k+2 = (k+2) + k \geq k+2$ (since $k \geq 0$). So $2^{k+1} \geq (k+1)+1$. By induction, true for all $n \geq 0$.

> **Investigate further:** For which n does equality $2^n = n+1$ actually hold?

8. Assuming $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$, show that $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$.

Answer: $\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$

Method: $k(k+2)+1 = k^2+2k+1 = (k+1)^2$, which cancels one factor of $(k+1)$ from the denominator.

> **Investigate further:** This is a telescoping sum: write $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$ and confirm the telescoping gives the same closed form directly.

9. Prove by induction that the sum of the interior angles of a convex n-gon is $(n-2) \times 180°$ for all $n \geq 3$.

Answer: Proof by induction

Method: Base case $n = 3$: a triangle has angle sum $180° = (3-2) \times 180°$. Holds.
Inductive step: assume a convex k-gon has angle sum $(k-2) \times 180°$. A convex $(k+1)$-gon can be split, by drawing one diagonal from a vertex, into a convex k-gon and a triangle, whose angle sums combine to give the full $(k+1)$-gon's angle sum: $(k-2) \times 180° + 180° = (k-1) \times 180° = ((k+1)-2) \times 180°$. By induction, true for all $n \geq 3$.

> **Investigate further:** Why does this diagonal-splitting argument require the $(k+1)$-gon to be convex?

10. A student wants to prove $P(n)$ for all $n \geq 1$ using base cases $P(1)$ and $P(2)$, with inductive step "$P(k)$ and $P(k+1)$ together imply $P(k+2)$" for $k \geq 1$. Explain why establishing *both* $P(1)$ and $P(2)$ (not just $P(1)$) is necessary here.

Answer: The inductive step requires two consecutive prior cases to fire; without $P(2)$ independently established, the chain can never produce $P(2)$ or anything depending on it.

Method: The step $P(k) \wedge P(k+1) \implies P(k+2)$ can only be applied once both $P(k)$ and $P(k+1)$ are already known. Starting from $P(1)$ alone gives no way to reach $P(2)$ — the recursion needs two "seeds", exactly analogous to how the Fibonacci recurrence needs two initial terms, not one.

> **Investigate further:** How many base cases would be needed for a step of the form "$P(k), P(k+1), P(k+2) \implies P(k+3)$"?

Section C

1. A flawed "proof" that everyone in any group shares a favourite integer.

Answer: B

Method: The overlap argument genuinely requires the two subgroups of size k (formed by removing different people from a $(k+1)$-person group) to share at least one common member, which needs $k \geq 2$. The transition from $P(1)$ to $P(2)$ uses $k = 1$, where the "two subgroups" are single individuals that share no member — so this specific step is invalid, and the whole chain never gets off the ground, regardless of how valid later steps might look in isolation.

> **Investigate further:** Confirm that the step from $P(2)$ to $P(3)$ (i.e. $k = 2$) would genuinely be valid in isolation, if only it could be reached.

2. Evaluate an inductive step for $n^3 - n$ divisible by 6 that appeals to "the general theory" instead of doing the algebra.

Answer: B

Method: A correct inductive step must actually derive $(k+1)^3 - (k+1)$ algebraically from $k^3 - k = 6m$: $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = (k^3 - k) + 3k^2 + 3k = 6m + 3k(k+1)$. Since $k(k+1)$ is a product of consecutive integers, it is even, so $3k(k+1)$ is divisible by 6, and the whole expression is $6m + 6(\text{integer})$, divisible by 6. The original "proof" skips this derivation entirely, merely asserting the conclusion.

> **Investigate further:** Complete the missing step: why is $k(k+1)$ always even?

3. Evaluate a base-case dispute for $n^2 > 3n + 4$.

Answer: B

Method: $n^2 - 3n - 4 = (n-4)(n+1)$, positive exactly when $n > 4$ (for positive n). So $P(4)$: $16 > 16$ is false (equality, not strict), but $P(5)$: $25 > 19$ is true. The inductive step holds for all $k \geq 5$: assuming $k^2 - 3k - 4 > 0$ (i.e. $P(k)$), we have $(k+1)^2 - 3(k+1) - 4 = k^2 + 2k + 1 - 3k - 3 - 4 = (k^2 - 3k - 4) + (2k - 2)$, a sum of a positive quantity (the inductive hypothesis) and another positive quantity ($2k - 2 \geq 8 > 0$ for $k \geq 5$), hence positive, giving $P(k+1)$. A false $P(n_0)$ at one specific point does not mean induction is impossible — it only means that particular n_0 cannot serve as the base case.

> **Investigate further:** Verify directly that $P(5) \implies P(6)$: check $36 > 22$.

4. Evaluate a weak-induction attempt to prove every $n \geq 2$ is a product of primes.

Answer: B

Method: The fatal gap: weak induction's hypothesis only concerns k, but factoring $k+1$ (when composite) requires information about its proper factors $a, b < k+1$, which are generally unrelated to k itself. Strong induction fixes this by assuming the claim for every integer from 2 to k, so that whichever factors a, b of $k+1$ arise, both are already covered by the (stronger) inductive hypothesis.

> **Investigate further:** Write out the corrected strong-induction version of this proof in full.

5. Evaluate whether a failed $P(1)$ means no base case exists for $\frac{1}{n+1} + \cdots + \frac{1}{2n} > \frac{1}{2}$.

Answer: B

Method: $P(1)$ gives exactly $\frac{1}{2}$, so $\frac{1}{2} > \frac{1}{2}$ is false — $n_0 = 1$ doesn't work. But $P(2)$: $\frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{1}{2}$, true. In fact this sum is increasing in n (each step replaces the smallest term with two smaller-but-larger-summing terms), so $n_0 = 2$ works as the base case.

> **Investigate further:** Show algebraically that the sum strictly increases when going from n to $n+1$.

6. Evaluate the "3p/5p postage" strong-induction step.

Answer: A

Method: Since every increment of 1 in the target amount can be achieved by adding one 3p stamp to a construction that is 3 less, the inductive step must reach back to $P(k-2)$, not $P(k)$ — this is why three consecutive base cases $(8, 9, 10)$ are needed to seed every residue class mod 3 before the "step back by 3" induction can run indefinitely.

> **Investigate further:** Which two-stamp combinations show that $11, 12, 13$ pence are also directly constructible, confirming the pattern from the base cases?

7. Evaluate a finite-case "proof" of the alternating Fibonacci sum identity.

Answer: B

Method: Direct check: $n = 1$: $F_1 = 1 = F_2$. $n = 2$: $F_1 + F_3 = 1 + 2 = 3 = F_4$. $n = 3$: $+F_5 = 3 + 5 = 8 = F_6$. $n = 4$: $+F_7 = 8+13 = 21 = F_8$. All hold, but this is still only 4 cases. The correct inductive step: assuming $F_1+\cdots+F_{2k-1} = F_{2k}$, add F_{2k+1} to both sides: $F_1+\cdots+F_{2k-1}+F_{2k+1} = F_{2k}+F_{2k+1} = F_{2k+2}$ (by the Fibonacci recurrence itself), which is exactly the $n = k + 1$ case.

> **Investigate further:** Why does this identity's inductive step require essentially no extra algebra, unlike most sum identities?

8. Which statement precisely captures the inductive step?

Answer: C

Method: The inductive step is itself a universally-quantified conditional statement: for every $k \geq 1$, IF $P(k)$ holds THEN $P(k+1)$ holds. Options A, B, D, and E each misstate either the quantifier (existential instead of universal, or missing entirely) or the logical connective.

> **Investigate further:** Which option (D) describes something logically much weaker than the actual inductive step, and why is it too weak to be useful?

Section D

1. $t = x + \frac{1}{x}$ integer > 2; define $t_n = x^n + \frac{1}{x^n}$. *(after BMO1 2015 Q4)*

Answer: (a) $t_{n+1} = t \cdot t_n - t_{n-1}$. (b) Proof by strong induction below.

Method: (a) $t \cdot t_n = \left(x + \frac{1}{x}\right)\left(x^n + \frac{1}{x^n}\right) = x^{n+1} + x^{1-n} + x^{n-1} + \frac{1}{x^{n+1}} = \left(x^{n+1} + \frac{1}{x^{n+1}}\right) + \left(x^{n-1} + \frac{1}{x^{n-1}}\right) = t_{n+1} + t_{n-1}$. Rearranging gives $t_{n+1} = t \cdot t_n - t_{n-1}$.
(b) Base cases: $t_0 = 2$ (integer) and $t_1 = t$ (integer, given). Strong inductive step: assume t_{n-1} and t_n are both integers for some $n \geq 1$. Since t is an integer and $t_{n+1} = t \cdot t_n - t_{n-1}$ is a sum/product of integers, t_{n+1} is also an integer. By strong induction, t_n is an integer for every positive integer n.

> **Investigate further:** Why does this proof genuinely need strong (two-step-back) induction rather than ordinary single-step induction?

2. Prove every positive integer is a sum of distinct charming integers (2 or 3^i5^j). *(after BMO1 2016*

Q6)

Answer: Proof by strong induction below, via the key gap lemma.

Method: *Key lemma*: if $c < c'$ are consecutive charming integers (no charming integer strictly between them), then $c' < 2c$. Spot-checking the sorted charming integers $1, 2, 3, 5, 9, 15, 25, 27, 45, 75, 81, \ldots$ confirms every consecutive ratio is below 2 (e.g. $9/5 = 1.8$, $15/9 \approx 1.67$, $27/25 = 1.08$); a full general proof requires bounding how far apart consecutive values of 3^i5^j can be, using that $5/3 < 2$ and $3 < 2 \times 2$ constrain how much any single-exponent increment can jump — this gap-bound is the genuine crux of the problem's difficulty, exactly why it carries full marks as a BMO1 question rather than being a one-line observation.
Base case $n = 1$: $1 = 3^05^0$ is itself charming, so it trivially "sums" to itself.
Strong inductive step: assume every positive integer less than n (for some $n \geq 2$) can be written as a sum of distinct charming integers. If n is itself charming, it trivially works. Otherwise, let c be the largest charming integer with $c < n$, and let c' be the next charming integer above c. Since c was chosen as the largest charming integer below n, we have $c' \geq n$; combined with the key lemma ($c' < 2c$), this gives $n \leq c' < 2c$, i.e. $n - c < c$. So $0 < n - c < c < n$, and by the strong inductive hypothesis, $n - c$ is a sum of distinct charming integers, each of which is $\leq n - c < c$ and therefore cannot equal c itself. Adding c to this sum gives n as a sum of distinct charming integers, completing the induction.

> **Investigate further:** Verify the base cases $n = 1, 2, 3, 4$ explicitly by hand, and check the gap lemma holds for the first ten charming integers listed above.

3. Prove Bernoulli's Inequality: $(1 + x)^n \geq 1 + nx$ for $x > -1$, n a positive integer.

Answer: Proof by induction below; the condition $x > -1$ is required when multiplying the inequality by $(1 + x)$.

Method: Base case $n = 1$: $(1 + x)^1 = 1 + x = 1 + 1 \cdot x$. Holds with equality.
Inductive step: assume $(1+x)^k \geq 1+kx$ for some $k \geq 1$. Since $x > -1$, we have $1+x > 0$, so multiplying both sides of the hypothesis by $(1 + x)$ *preserves* the inequality direction: $(1 + x)^{k+1} \geq (1 + kx)(1 + x) = 1+x+kx+kx^2 = 1+(k+1)x+kx^2$. Since $kx^2 \geq 0$, this gives $(1+x)^{k+1} \geq 1+(k+1)x+kx^2 \geq 1+(k+1)x$. By induction, true for all positive integers n.

> **Investigate further:** What would go wrong with the argument if $x < -1$ were allowed (i.e. $1 + x < 0$)?

4. $a_1 = 2, a_{n+1} = a_n^2 - a_n + 1$. Prove $a_1, \ldots, a_n$ pairwise coprime.

Answer: Proof by induction below (via the lemma $a_n - 1 = \prod_{i=1}^{n-1} a_i$).

Method: *Lemma*: $a_n - 1 = \prod_{i=1}^{n-1} a_i$ for all $n \geq 2$, proved by induction on n.
Base case $n = 2$: $a_2 - 1 = (2^2 - 2 + 1) - 1 = 3 - 1 = 2 = a_1$. Holds.
Inductive step: assume $a_k - 1 = \prod_{i=1}^{k-1} a_i$. From the recurrence, $a_{k+1} - 1 = a_k^2 - a_k = a_k(a_k - 1) = a_k \prod_{i=1}^{k-1} a_i = \prod_{i=1}^{k} a_i$. Lemma holds for all $n \geq 2$ by induction.
Pairwise coprimality: take any $i < j \leq n$. By the lemma, $a_j - 1 = \prod_{m=1}^{j-1} a_m$, a product that includes the factor a_i (since $i \leq j-1$). So $a_i \mid (a_j - 1)$, i.e. $a_j \equiv 1 \pmod{a_i}$. Hence $\gcd(a_i, a_j) = \gcd(a_i, a_j - 1 + 1) = \gcd(a_i, 1) = 1$ (using that $a_j - 1$ is a multiple of a_i). Since i, j were arbitrary, $a_1, \ldots, a_n$ are pairwise coprime.

> **Investigate further:** Compute a_1, a_2, a_3, a_4 explicitly and verify they are pairwise coprime by direct inspection.

5. Prove $\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$ for all integers $n \geq 2$.

Answer: Proof by induction below.

Method: Base case $n = 2$: LHS $= 1 + \frac{1}{4} = \frac{5}{4}$. RHS $= 2 - \frac{1}{2} = \frac{3}{2}$. Since $\frac{5}{4} < \frac{3}{2}$, holds.
Inductive step: assume $\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$ for some $n \geq 2$. Then $\sum_{k=1}^{n+1} \frac{1}{k^2} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$. It suffices to show $-\frac{1}{n} + \frac{1}{(n+1)^2} \leq -\frac{1}{n+1}$, i.e. $\frac{1}{(n+1)^2} \leq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. This holds because $(n+1)^2 > n(n+1)$ (as $n+1 > n$), so $\frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$, strictly. Combining, $\sum_{k=1}^{n+1} \frac{1}{k^2} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n} + \frac{1}{n(n+1)} = 2 - \frac{1}{n+1}$. By induction, true for all $n \geq 2$.

> **Investigate further:** What does this inequality suggest about the value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ as $n \to \infty$?