# Daily Logic Drill #5 — Answers & Investigations

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| Section | Topic | Questions | Target time |
|---|---|---|---|
| A | Rapid Recognition | 10 | 2 min 30 sec |
| B | Manipulation Drills | 10 | 8 min |
| C | Substitution & Structure | 8 | 10 min |
| D | Challenge | 5 | 15 min |

*New toolkit today: Spot the Flaw · Common Fallacies · Proof Critique · Valid vs. Invalid Arguments.*

## Section A

**1.** Name the fallacy: "If $n$ is divisible by 6, then $n$ is divisible by 3. $n = 15$ is divisible by 3. Therefore $n = 15$ is divisible by 6."

**Answer:** Affirming the consequent

**Method:** The argument has the form "$P \implies Q$, $Q$, therefore $P$", which is invalid: knowing $Q$ is true does not establish $P$. Indeed 15 is divisible by 3 but not by 6.

> **Investigate further:** Write the valid rule that this fallacy is often confused with (modus ponens).

**2.** Name the fallacy: "If $x > 10$, then $x > 0$. Suppose $x \leq 10$. Therefore $x \leq 0$."

**Answer:** Denying the antecedent

**Method:** The argument has the form "$P \implies Q$, $\neg P$, therefore $\neg Q$", which is invalid. E.g. $x = 5$ satisfies $x \leq 10$ but not $x \leq 0$.

> **Investigate further:** What is the valid rule that instead concludes from $\neg Q$ to $\neg P$?

**3.** True or false: "If a proof of $P \implies Q$ is shown to be flawed, then $P \implies Q$ must be false."

**Answer:** False

**Method:** A flawed proof only invalidates that particular argument; $P \implies Q$ may still be true and simply require a different, correct proof.

> **Investigate further:** Give an example of a true statement that is easy to state but whose standard proof most students get wrong on a first attempt.

**4.** A 6-line "proof" is flawed only on line 4. How many of the six lines can you actually trust?

**Answer:** 3 (lines 1–3)

**Method:** Once line 4 is wrong, every subsequent line (5 and 6) depends on it and cannot be trusted either, even if the algebra within them is locally correct.

> **Investigate further:** Could line 6 coincidentally still be a true statement despite depending on a flawed line 4?

**5.** True or false: "Squaring both sides of an equation is a reversible step that can never introduce a new, extraneous solution."

**Answer:** False

**Method:** Squaring is not injective on negative numbers: e.g. $x = -2$ does not satisfy $x = 2$, but squaring both sides gives $x^2 = 4$, which is also satisfied by $x = 2$.

> **Investigate further:** Name another common "operation" on an equation that can introduce extraneous solutions.

**6.** A "proof" concludes $0 = 1$. What does this tell you about the argument (assuming it is not a contradiction proof)?

**Answer:** At least one line of the argument contains an error.

**Method:** $0 = 1$ is false. A valid argument cannot derive a false conclusion from true premises, so an error must exist somewhere in the chain of reasoning.

> **Investigate further:** How would your answer change if this occurred inside a proof by contradiction?

**7.** True or false: "Verifying a universal claim $P(n)$ for $n = 1, 2, \ldots, 10$ constitutes a full proof that $P(n)$ holds for all positive integers $n$."

**Answer:** False

**Method:** Finite verification never proves a $\forall n$ claim; a counterexample could exist at $n = 11$ or beyond.

> **Investigate further:** What proof technique specifically is designed to close this gap for claims about all positive integers?

**8.** What is the flaw in dividing both sides of $ax = bx$ by $x$ to conclude $a = b$?

**Answer:** The step is invalid if $x = 0$.

**Method:** If $x = 0$, $ax = bx$ becomes $0 = 0$, which is true for any $a, b$ — so $a = b$ need not hold. Dividing by a possibly-zero quantity can lose solutions/cases.

> **Investigate further:** What should be written instead to handle the $x = 0$ case correctly?

**9.** True or false: "In a proof by contradiction, the final derived statement must be a strict logical contradiction (such as $0 = 1$), not merely an unusual-looking result."

**Answer:** True

**Method:** A valid contradiction proof requires deriving something that is genuinely impossible (a statement and its negation both holding, or a false arithmetic identity) — an "unusual" but not actually impossible result does not complete the proof.

> **Investigate further:** Give an example of a result that looks surprising but is not actually a contradiction.

**10.** What is wrong with concluding from $\dfrac{a}{b} = \dfrac{c}{d}$ that $a = c$ and $b = d$?

**Answer:** Equal fractions do not require equal numerators and denominators separately.

**Method:** E.g. $\frac{1}{2} = \frac{2}{4}$, yet $1 \neq 2$ and $2 \neq 4$. Equality of fractions only guarantees the cross-multiplied products match: $ad = bc$.

> **Investigate further:** Under what extra condition on $a, b, c, d$ would $\frac{a}{b} = \frac{c}{d}$ actually force $a = c, b = d$?

## Section B

**1.** A student attempts to prove: "For all integers $n$, if $n$ is odd, then $n^2$ is odd."
Line 1: Suppose $n$ is odd, so $n = 2k + 1$ for some integer $k$.
Line 2: Then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
Line 3: Since $2k^2 + 2k$ is an integer, $n^2$ is odd.
Is this proof valid? Justify your answer.

**Answer:** Yes, fully valid.

**Method:** Every line follows correctly: the algebra in Line 2 is correct ($(2k + 1)^2 = 4k^2 + 4k + 1$), and Line 3 correctly identifies the result as $2 \times \text{(integer)} + 1$, the definition of odd. There is no flaw.

> **Investigate further:** Is this a direct proof or a proof by contrapositive of some related statement?

**2.** Find the flaw: "Claim: If $x^2 = 3x$, then $x = 3$." Proof: Divide both sides of $x^2 = 3x$ by $x$ to get $x = 3$.

**Answer:** Dividing by $x$ is invalid when $x = 0$.

**Method:** $x^2 = 3x \iff x(x - 3) = 0 \iff x = 0$ or $x = 3$. Dividing by $x$ silently assumes $x \neq 0$ and loses the solution $x = 0$, which also satisfies the original equation.

> **Investigate further:** Rewrite the claim correctly, including both solutions.

**3.** Find the flaw: "Claim: The only real solution to $\sqrt{x + 7} = x - 5$ is $x = 9$." Proof: Squaring both sides gives $x + 7 = x^2 - 10x + 25$, so $x^2 - 11x + 18 = 0$, so $(x - 9)(x - 2) = 0$, giving $x = 9$ or $x = 2$.

**Answer:** $x = 2$ is an extraneous root introduced by squaring; it must be checked against and rejected from the original equation.

**Method:** Checking $x = 2$ in the original equation: $\sqrt{2 + 7} = \sqrt{9} = 3$, but $x - 5 = 2 - 5 = -3$. Since $3 \neq -3$, $x = 2$ does not satisfy the original equation. Squaring introduced this extraneous solution; only $x = 9$ (which does check: $\sqrt{16} = 4 = 9 - 5$) is valid.

> **Investigate further:** Why does squaring specifically introduce solutions where the two sides had opposite sign?

**4.** A student argues: For every integer $n$, $6\left(\dfrac{4n+1}{2} - \dfrac{n-3}{2}\right)$ is a multiple of 3. *(after TMUA 2020 Paper 2 Q3)*
Line 1: $6\left(\dfrac{4n+1}{2} - \dfrac{n-3}{2}\right) = 3\left(2\left(\dfrac{4n+1}{2} - \dfrac{n-3}{2}\right)\right)$.
Line 2: $= 3(4n + 1 - n - 3)$.
Line 3: $= 3(3n - 2)$.
Line 4: which is always a multiple of 3.
On which line does the argument first contain an error, and what exactly is that error?

**Answer:** Line 2

**Method:** Correctly distributing, $2\left(\frac{4n+1}{2} - \frac{n-3}{2}\right) = (4n + 1) - (n - 3) = 4n + 1 - n + 3 = 3n + 4$, not $4n + 1 - n - 3$ as written. The sign of $-3$ was not flipped when distributing the subtraction across the bracket $(n-3)$. (Interestingly, since Line 1's outer factor of 3 is untouched by this error, the final conclusion — a multiple of 3 — remains true regardless; the argument is still invalid even though its conclusion happens to be correct.)

> **Investigate further:** Compute the genuinely correct simplified form, $3(3n+4)$, and confirm it is still a multiple of 3.

**5.** Find the flaw: "Claim: For every integer $n$, $n^2+n+2$ is even." Proof: If $n$ is even, $n=2k$, so $n^2+n+2 = 4k^2+2k+2 = 2(2k^2+k+1)$, which is even. Hence $n^2+n+2$ is even for every integer $n$.

**Answer:** The proof only checks the even case; the odd case is never addressed.

**Method:** To prove a claim for "every integer $n$", both parities must be checked. (The claim does happen to also hold for odd $n$: $n = 2k+1 \implies n^2+n+2 = 4k^2+4k+1+2k+1+2 = 4k^2+6k+4 = 2(2k^2+3k+2)$, even — but the proof as given never establishes this.)

> **Investigate further:** Complete the missing odd case explicitly.

**6.** Find the flaw: "Claim: For real $x$, $x^2 = 9 \iff x = 3$." Proof: If $x = 3$, then $x^2 = 9$. This proves the biconditional.

**Answer:** Only the $\Longleftarrow$ direction ($x = 3 \implies x^2 = 9$) was shown; the $\Longrightarrow$ direction ($x^2 = 9 \implies x = 3$) is false, since $x = -3$ also satisfies $x^2 = 9$.

**Method:** A biconditional requires both directions to be proved. The correct statement is $x^2 = 9 \iff x = 3$ or $x = -3$.

> **Investigate further:** What is the correct biconditional relating $x^2 = 9$ to a single equation in $x$?

**7.** Find the flaw: "Claim: For every positive integer $n$, the $n$-th Fibonacci number $F_n$ (with $F_1 = F_2 = 1$) is either equal to 1 or prime." Proof: $F_3 = 2$, $F_4 = 3$, $F_5 = 5$ — all prime. Since the pattern holds for $n = 3, 4, 5$, the claim is proved. What is the smallest counterexample?

**Answer:** Checking finitely many cases does not prove a universal claim; $F_6 = 8 = 2^3$ is the smallest counterexample.

**Method:** $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8$. $F_6 = 8$ is neither equal to 1 nor prime ($8 = 2 \times 4$), disproving the claim. Three verified cases give no logical guarantee about the fourth.

> **Investigate further:** Is $F_n$ prime for infinitely many $n$? This remains an open problem in general.

**8.** A student wants to prove: "If $n$ is not divisible by 3, then $n^2$ is not divisible by 3." Instead, they show: "If $n$ is divisible by 3, then $n^2$ is divisible by 3." Explain why this does not prove the original claim.

**Answer:** The statement shown is neither the original claim nor its contrapositive — it proves nothing about the original.

**Method:** Original claim: $\neg A \implies \neg B$ where $A$: "$3 \mid n$", $B$: "$3 \mid n^2$". Its valid contrapositive is $B \implies A$ ("if $3 \mid n^2$ then $3 \mid n$"). The student instead proved $A \implies B$, a different (though also true) statement, which establishes neither the original claim nor its contrapositive.

> **Investigate further:** State the actual contrapositive of the original claim, and outline how you would prove it.

**9.** Find the flaw: "Claim: For all integers $a, b$, if $a$ and $b$ are both even, then $a+b$ is even." Proof: Since $a+b$ is even, $a+b=2m$ for some integer $m$. Since $a, b$ are even, $a=2j, b=2k$, so $2j+2k=2m$, i.e. $j+k=m$, which is true since $m$ is an integer. Hence $a+b$ is even.

**Answer:** Circular reasoning: the proof begins by assuming $a+b$ is even (the very thing to be proved) instead of deriving it.

**Method:** A correct proof starts from $a=2j, b=2k$ and derives $a+b=2j+2k=2(j+k)$, which is even because $j+k$ is an integer — it never needs to assume $a+b=2m$ up front.

> **Investigate further:** Rewrite this as a correct, non-circular direct proof.

**10.** A student is told: "It is not the case that every student in the class passed the exam." They conclude: "Therefore, every student in the class failed the exam." What is wrong with this conclusion?

**Answer:** The correct negation of "every student passed" is "at least one student did not pass", not "every student failed".

**Method:** $\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$, not $\forall x\, \neg P(x)$. Some students may have passed while at least one did not.

> **Investigate further:** Which named error (from Section A of an earlier drill) does this conclusion resemble?

## Section C

**1.** Consider the claim: "For every positive integer $n$, $2n^2+2n+1$ is prime." *(after TMUA 2022 Paper 2 Q7)*

**Answer:** D

**Method:** Lines I–III are correct: the discriminant of $2x^2+2x+1$ is indeed $-4$, so the polynomial has no real (let alone integer linear) factorisation. But Line IV commits a category error: a polynomial being irreducible over $\mathbb{Z}[x]$ says nothing about whether its integer OUTPUT VALUES are prime. At $n=3$: $2(9)+6+1=25=5^2$, composite, disproving the original claim.

> **Investigate further:** Find another value of $n$ for which $2n^2+2n+1$ is composite.

**2.** Let $P$: "$x^2-5x+6=0$" and $Q$: "$x=2$". A student argues $Q \implies P$ is true and concludes $P \iff Q$.

**Answer:** B

**Method:** $x^2-5x+6=(x-2)(x-3)=0$, so $P$ is satisfied by $x=2$ or $x=3$. $Q \implies P$ is indeed true, but $P \implies Q$ is false since $x=3$ satisfies $P$ without satisfying $Q$. Asserting $P \implies Q$ merely because the statements "seem related" is the flaw.

> **Investigate further:** What is the correct relationship (necessary/sufficient) between $P$ and $Q$?

**3.** A student attempts to prove: "For every integer $n$, $n^2+n$ is divisible by 2", checking only the even case.

**Answer:** B

**Method:** $n^2+n=n(n+1)$ is a product of two consecutive integers, so it is even regardless of the parity of $n$ — but the given proof only demonstrates this for even $n$ and never addresses odd $n$, so as written it is incomplete.

> **Investigate further:** Complete the missing odd case: for $n = 2k+1$, show $n^2 + n$ is even.

**4.** A student attempts to prove: "No three consecutive odd positive integers can all be prime." *(after TMUA 2023 Paper 2 Q4)*

**Answer:** C

**Method:** Lines I and II are correct: among $n-2, n, n+2$, exactly one is always divisible by 3. Line III is the flaw — being a positive multiple of 3 only implies compositeness if the multiple is *greater than* 3; the multiple could equal 3 itself, which is prime. Exactly this happens at $n = 5$: the three consecutive odd integers $3, 5, 7$ are all prime.

> **Investigate further:** Are there any other counterexamples to the original claim besides $3, 5, 7$?

**5.** A student "proves" $\forall n\, \exists m\, (m > n)$ and concludes $\exists M\, \forall n\, (M > n)$.

**Answer:** B

**Method:** Swapping the order of a universal and existential quantifier is not a valid logical step in general. $\forall n\, \exists m\, (m > n)$ says every $n$ has SOME larger $m$ (true, e.g. $m = n+1$); $\exists M\, \forall n\, (M > n)$ says a SINGLE $M$ exceeds every $n$ simultaneously, which is false (no integer is larger than every integer).

> **Investigate further:** Give a general example of a true $\forall\exists$ statement whose $\exists\forall$ swap is false.

**6.** A student "confirms" the converse of "rhombus $\implies$ perpendicular diagonals" using only the example of a square.

**Answer:** B

**Method:** One example (the square) supports the converse but cannot prove a universal statement. A kite has perpendicular diagonals but is generally not a rhombus, disproving the converse.

> **Investigate further:** Sketch a kite that is not a rhombus, and confirm its diagonals are perpendicular.

**7.** A student disproves $(*)$: "For every pair of real numbers $x, y$, $|x+y| < |x| + |y|$" using the pair $x = -2, y = -5$. *(after TMUA 2020 Paper 2 Q8)*

**Answer:** D

**Method:** This is a trap: the student's reasoning is actually completely correct. $(*)$ as stated uses strict $<$, and at $x = -2, y = -5$ (both negative, hence same sign), $|x+y| = 7 = |x| + |y|$, so $7 < 7$ is false — a fully valid single counterexample to a universal claim. No further pairs need be checked to reject $(*)$.

> **Investigate further:** For which pairs $(x, y)$ does $|x+y| = |x| + |y|$ hold with equality?

**8.** A student "verifies" "four equal sides $\iff$ square" by checking only that every square has four equal sides.

**Answer:** B

**Method:** This checks only necessity (square $\implies$ four equal sides). Sufficiency (four equal sides $\implies$ square) is false: a non-square rhombus has four equal sides but is not a square.

> **Investigate further:** What additional condition, combined with "four equal sides", would give a genuine necessary and sufficient characterisation of a square?

## Section D

**1.** A flawed attempt to prove: if $s^2 \geq 4p$ then there exist real $u, v$ with $u + v = s, uv = p$. *(after TMUA 2021 Paper 2 Q11)*

**Answer:** B

**Method:** Every algebraic step (II, III) is correct, but the logical direction is backward. The student assumed $u, v$ exist and derived $s^2 \geq 4p$ from that assumption — this proves the CONVERSE of the theorem, not the theorem itself. A correct proof constructs $u, v$ directly: let $u, v$ be the two roots of $t^2 - st + p = 0$; since the discriminant $s^2 - 4p \geq 0$, these roots are real, and by Vieta's formulas $u + v = s, uv = p$.

> **Investigate further:** Why does the discriminant condition $s^2 - 4p \geq 0$ guarantee the roots of $t^2 - st + p = 0$ are real?

**2.** A student claims "$n^2 - n + 11$ prime $\implies$ $n^2 + n + 11$ prime" for all positive $n$, verified for $n = 1, \ldots, 9$.

**Answer:** (a) Finite verification is not a universal proof. (b) $n = 10$.

**Method:** (a) Checking 9 cases gives no guarantee about $n = 10$ or beyond — a single unchecked value could be a counterexample.
(b) At $n = 10$: $n^2 - n + 11 = 100 - 10 + 11 = 101$, which is prime (premise true). But $n^2 + n + 11 = 100 + 10 + 11 = 121 = 11^2$, which is NOT prime (conclusion false). So the implication fails at $n = 10$, the smallest such value since $n = 1, \ldots, 9$ were verified to hold.

> **Investigate further:** Verify that $n^2 - n + 11$ is prime for $n = 1, \ldots, 9$ by computing the first few values.

**3.** Evaluate a contradiction-style proof that "$ab$ even $\implies$ $a$ even or $b$ even".

**Answer:** D) I and II only

**Method:** The algebra $(2j + 1)(2k + 1) = 4jk + 2j + 2k + 1 = 2(2jk + j + k) + 1$ is correct, so III is false. Statement I is true: assuming the negation ($a, b$ both odd) and deriving a contradiction ($ab$ odd, contradicting $ab$ even) is a valid contradiction proof. Statement II is also true: the derivation "$a, b$ both odd $\implies$ $ab$ odd" is exactly the contrapositive of the original claim (contrapositive of $ab$ even $\implies$ ($a$ even or $b$ even) is $\neg$($a$ even or $b$ even) $\implies$ $\neg$($ab$ even), i.e. "$a, b$ both odd $\implies$ $ab$ odd"), proved here directly without needing the "suppose for contradiction" framing at all.

> **Investigate further:** Rewrite this proof in pure contrapositive form, removing the contradiction framing entirely.

**4.** Evaluate the "every odd prime $> 3$ has the form $6k \pm 1$" argument.

**Answer:** B

**Method:** Every line is correct: $6k, 6k+2, 6k+4$ are even (Line 2, correctly dismissed); $6k+3 = 3(2k+1)$ is a multiple of 3 exceeding 3 whenever $k \geq 1$, and the case $k = 0$ giving exactly 3 is excluded by the claim's own hypothesis "greater than 3" (Line 3, correctly handled). The proof only ever claimed to show that primes $> 3$ MUST have the form $6k \pm 1$ (necessity) — it never claimed every number of that form IS prime (sufficiency), and the original claim, read carefully, only asserts the former. So there is no actual gap.

> **Investigate further:** Give an example of a number of the form $6k + 1$ that is not prime, to confirm sufficiency genuinely fails.

**5.** Evaluate: "For all $n \geq 2$, $n^2 - 1$ is composite", proved via $n^2 - 1 = (n - 1)(n + 1)$.

**Answer:** (a) Line 3. (b) $n = 2$; and $n = 2$ is the unique exception.

**Method:** (a) Line 3 is the gap: a product of two integers each "at least 1" is not necessarily composite — compositeness requires BOTH factors to be at least 2. Line 2 only guarantees $n-1 \geq 1$, not $n-1 \geq 2$. (b) At $n = 2$: $n^2 - 1 = 3 = 1 \times 3$. The factor of 1 does not make 3 composite — 3 is prime, so the conclusion genuinely fails here. *Proof that $n = 2$ is the only failure*: for every $n \geq 3$, $n - 1 \geq 2$ and $n + 1 \geq 4$, so $n^2 - 1 = (n - 1)(n + 1)$ is a product of two integers both at least 2, which is precisely the definition of composite — no case analysis or further checking is needed once $n \geq 3$, since the factorisation itself, together with $n-1 \geq 2$, is a fully general and complete argument covering every such $n$ simultaneously.

> **Investigate further:** Why does the same style of argument fail to show $n^2 - 4$ is composite for all $n \geq 3$? (Check $n = 3$: $n^2 - 4 = 5$.)