Daily Logic Drill #3 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Direct Proof · Proof by Contrapositive.

Section A Rapid Recognition

1. Prove directly: "If n is even, then n^2 is even."

Answer: Proved.

Method: Let $n = 2k$ for an integer k. Then $n^2 = 4k^2 = 2(2k^2)$, which is twice an integer, hence even.

> **Investigate further:** This exact fact has already been *used* in Days 1–2; today's toolkit is about formally justifying it. Compare with A2's odd case and A7's contrapositive route to the reverse implication.

2. Prove directly: "If n is odd, then n^2 is odd."

Answer: Proved.

Method: Let $n = 2k+1$ for an integer k. Then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, one more than an even number, hence odd.

> **Investigate further:** This is exactly the contrapositive of "n^2 even $\Rightarrow$ n even" — A10 reuses this proof unchanged under that framing.

3. Prove directly: "If a and b are both even, then $a + b$ is even."

Answer: Proved.

Method: Let $a = 2j$ and $b = 2k$ for integers j, k. Then $a + b = 2j + 2k = 2(j + k)$, which is even.

> **Investigate further:** The same substitution-and-factor pattern as A1 — write each hypothesis as "$2\times$(something)", combine, and factor out the 2 again.

4. State the contrapositive of "If n is a multiple of 6, then n is even." Prove the contrapositive directly.

Answer: Contrapositive: "If n is odd, then n is not a multiple of 6." Proved.

Method: If n is odd, then n is not divisible by 2. Since every multiple of 6 is divisible by 2 (as $6 = 2 \times 3$), n cannot be a multiple of 6.

> **Investigate further:** Proving the contrapositive here is a one-line argument; proving the original directly would require the same reasoning run in reverse (multiple of 6 $\Rightarrow$ divisible by 2 $\Rightarrow$ even), which is no harder — this pair is a gentle first example, not yet a case where contrapositive is clearly the better route (see B3 for that).

5. Prove directly: "If a and b are both odd, then ab is odd."

> **Answer:** Proved.

Method: Let $a = 2j+1$ and $b = 2k+1$. Then $ab = (2j+1)(2k+1) = 4jk+2j+2k+1 = 2(2jk+j+k)+1$, one more than an even number, hence odd.

> **Investigate further:** Multiplying out two "odd" expressions always leaves a single stray $+1$ outside the bracket — the same signature as A2's proof.

6. True or false: "To prove 'if P then Q' by contrapositive, you should assume Q is false and show P is false."

> **Answer:** True.

Method: This is exactly the definition: proving the contrapositive "if not Q then not P" means assuming not Q (i.e. Q false) and deriving not P (i.e. P false).

> **Investigate further:** Contrast with proof by contradiction (a later toolkit item), which instead assumes P true and Q false *simultaneously* and derives an impossibility — a related but distinct setup.

7. Prove by contrapositive: "If n^2 is odd, then n is odd."

> **Answer:** Proved via the contrapositive "if n is even, then n^2 is even" (A1).

Method: The contrapositive of "n^2 odd $\Rightarrow n$ odd" is "n even $\Rightarrow n^2$ even", which is exactly A1, already proved. Since the contrapositive is true and shares the original's truth value, the original is true too.

> **Investigate further:** This is the day's first "reuse an earlier proof under a new name" moment — the whole payoff of having both techniques is not re-deriving facts you already have.

8. True or false: "A proof by contrapositive is a different technique from proving the converse."

> **Answer:** True.

Method: Proof by contrapositive establishes the *same* statement ("if P then Q") via its logically equivalent form "if not Q then not P". Proving the converse establishes a *different* statement ("if Q then P") entirely, which may or may not be true independent of the original.

> **Investigate further:** Confusing these two is the single most common TMUA Paper 2 trap (Day 1–2's whole converse toolkit exists to inoculate against it) — a proof that quietly proves the converse instead of the contrapositive has proved the wrong thing.

9. Prove directly: "If $x > 1$, then $x^2 > 1$."

> **Answer:** Proved.

Method: Since $x > 1 > 0$, both sides of $x > 1$ are positive, so multiplying the inequality by x (which is positive, preserving the direction) gives $x^2 > x$. Since $x > 1$, combining gives $x^2 > x > 1$, so $x^2 > 1$.

> **Investigate further:** Multiplying an inequality by a variable is only safe when that variable's sign is known — here $x > 1$ guarantees $x > 0$, making the step legitimate.

10. Prove by contrapositive: "If $x^2 \leq 1$, then $x \leq 1$."

Answer: Proved via the contrapositive "if $x > 1$, then $x^2 > 1$" (A9).

Method: The contrapositive of "$x^2 \leq 1 \Rightarrow x \leq 1$" is "$x > 1 \Rightarrow x^2 > 1$", which is exactly A9, already proved.

> **Investigate further:** A9/A10 is the cleanest possible illustration of the toolkit: the "contrapositive" version was already sitting there as A9, needing no new work at all.

Section B Manipulation Drills

1. Prove directly: "If n is a multiple of 4, then n^2 is a multiple of 16."

Answer: Proved.

Method: Let $n = 4k$. Then $n^2 = 16k^2$, manifestly a multiple of 16.

> **Investigate further:** B9 proves the contrapositive of this exact statement — compare the two write-ups.

2. Prove by contrapositive: "If n^2 is not a multiple of 3, then n is not a multiple of 3."

Answer: Proved via the contrapositive "if n is a multiple of 3, then n^2 is a multiple of 3".

Method: The contrapositive of "$3 \nmid n^2 \Rightarrow 3 \nmid n$" is "$3 \mid n \Rightarrow 3 \mid n^2$", proved directly: $n = 3k \Rightarrow n^2 = 9k^2 = 3(3k^2)$, a multiple of 3.

> **Investigate further:** This is the easier half of the classical "3 prime $\Rightarrow (3 \mid n^2 \iff 3 \mid n)$" fact — the harder half (that $3 \mid n^2$ genuinely forces $3 \mid n$) needs primality, not just algebra.

3. Which proof route is more natural for "If n^2 is even, then n is even" — direct or contrapositive? Prove it using that route.

Answer: Contrapositive is more natural. Proved via "if n is odd, then n^2 is odd" (A2).

Method: A direct proof would have to start from "n^2 is even" and somehow extract a factor of 2 from n itself — not straightforward, since $n^2 = 2m$ doesn't obviously hand you $n = 2k$. The contrapositive route sidesteps this entirely: assume n odd, and A2 already shows n^2 is then odd, which is what's needed.

> **Investigate further:** This is the general signal for reaching for contrapositive: when the hypothesis is a claim about a *derived* quantity (n^2) and the conclusion is about the *original* quantity (n), working backwards from the conclusion's negation is usually cleaner than working forwards from the hypothesis.

4. Prove directly: "If $a > b > 0$, then $a^2 > b^2$."

Answer: Proved.

Method: Since $a > b > 0$, multiply $a > b$ by a (positive, so direction preserved) to get $a^2 > ab$. Multiply $a > b$ by b (positive) to get $ab > b^2$. Chaining: $a^2 > ab > b^2$, so $a^2 > b^2$.

> **Investigate further:** This is the two-variable generalisation of A9's single-variable argument — the same "multiply by a known-positive quantity" move, applied twice.

5. Prove by contrapositive: "If $x^2 - 4x + 3 \neq 0$, then $x \neq 1$ and $x \neq 3$."

Answer: Proved via the contrapositive "if $x = 1$ or $x = 3$, then $x^2 - 4x + 3 = 0$".

Method: At $x = 1$: $1 - 4 + 3 = 0$. At $x = 3$: $9 - 12 + 3 = 0$. Both cases give $x^2 - 4x + 3 = 0$, confirming the contrapositive directly by substitution.

> **Investigate further:** The contrapositive here trades an abstract "$\neq$" claim for two concrete substitutions — exactly the kind of simplification contrapositive is good for whenever the conclusion's negation is a finite, checkable list of cases.

6. Prove directly: "If n is a multiple of 3, then n^3 is a multiple of 27."

Answer: Proved.

Method: Let $n = 3k$. Then $n^3 = 27k^3$, manifestly a multiple of 27.

> **Investigate further:** The same one-line substitution pattern as B1, now with a cube instead of a square — $d \mid n \Rightarrow d^m \mid n^m$ works identically for any power m.

7. State and prove the contrapositive of "If xy is irrational, then x is irrational or y is irrational."

Answer: Contrapositive: "If x is rational and y is rational, then xy is rational." Proved.

Method: Negate both clauses and swap: "irrational or" negates (De Morgan) to "rational and". If $x = \frac{p}{q}$ and $y = \frac{r}{s}$ for integers p, q, r, s with $q, s \neq 0$, then $xy = \frac{pr}{qs}$, a ratio of integers with nonzero denominator, hence rational.

> **Investigate further:** The contrapositive here is dramatically easier than the original — rationals are closed under multiplication by definition, while the original statement would require reasoning about irrationals directly.

8. Prove directly: "If n ends in the digit 5, then n^2 ends in the digit 5."

Answer: Proved.

Method: If n ends in 5, write $n = 10k + 5$. Then $n^2 = 100k^2 + 100k + 25 = 100(k^2 + k) + 25$. Since $100(k^2 + k)$ contributes only to digits beyond the tens place, the last two digits of n^2 are exactly 25, so in particular the last digit is 5.

> **Investigate further:** This is a place-value argument, not a parity one — the technique (isolate the part of the expansion that affects only the digits you care about) generalises to any "last digit(s)" claim.

9. Prove by contrapositive: "If n is not a multiple of 4, then n^2 is not a multiple of 16."

Answer: Proved via the contrapositive "if n^2 is a multiple of 16, then n is a multiple of 4".

Method: This is more delicate than it looks: the direct converse-flavoured claim "$16 \mid n^2 \Rightarrow 4 \mid n$" is true, but only because $16 = 4^2$ is a perfect square matching the target exponent exactly (unlike, e.g., n^2 mult. of $4 \Rightarrow n$ mult. of 4, which is false, witnessed by $n = 6$). Writing $n = 4q + s$ for $s \in \{0, 1, 2, 3\}$ and checking each residue confirms only $s = 0$ gives $n^2 \equiv 0 \pmod{16}$.

> **Investigate further:** Contrast directly with B1: B1 shows $4 \mid n \Rightarrow 16 \mid n^2$ (always safe, any prime-power target), while this question's underlying fact ($16 \mid n^2 \Rightarrow 4 \mid n$) needs the exponents to match up exactly — a subtlety Day 1's C1/C8 pair first surfaced.

10. True or false: "Every proof by contrapositive can be rewritten as a direct proof of the original statement."

Answer: False.

Method: A proof by contrapositive establishes "if not Q then not P" by direct reasoning *about that statement*; it does not itself provide a chain of direct deductions starting from P and ending at Q. The two are different argument structures that happen to establish the same true statement — B3 is a concrete example where no natural direct argument was found at all.

> **Investigate further:** This is a genuinely debatable point in philosophy-of-proof circles, but for TMUA purposes the practical takeaway is simpler: contrapositive and direct proof are two distinct tools in the kit, and reaching for the wrong one can turn an easy problem into a hard one (B3) or vice versa.

Section C Substitution & Structure

1. For non-zero real numbers x, y, which one of the following is sufficient to conclude that $x < y$?

Answer: E) $x^5 < y^5$.

Method: The fifth power is a strictly increasing function on the reals (its rate of change $5t^4$ is never negative, and is zero only at the single point $t = 0$, which is not enough to ever reverse direction), so $x^5 < y^5 \iff x < y$ — option E is sufficient. Options A, B (x^2, y^2) fail: $x = 1, y = -2$ gives $x^2 = 1 < 4 = y^2$ (satisfies A's hypothesis) but $x < y$ is false $(1 < -2$ false). Options C, D (absolute value) fail via the same instance: $x = 1, y = -2$ gives $|x| = 1 < 2 = |y|$ (satisfies C's hypothesis) but $x < y$ is false. Option F $(y^5 < x^5)$ is the reverse of the true relation.

> **Investigate further:** Odd powers preserve order in both directions because they're strictly increasing across all of $\mathbb{R}$; squaring and absolute value both fold negative and positive inputs onto the same output range, losing exactly the sign information that determines order — the single fact underlying every "which operation is safe to apply to an inequality" question.

2. Prove by contrapositive: "If n^3 is even, then n is even."

Answer: Proved via the contrapositive "if n is odd, then n^3 is odd".

Method: Let $n = 2k + 1$. Then $n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$, one more than an even number, hence odd.

> **Investigate further:** The same pattern extends to any odd power: "if n^m is even then n is even" for any positive integer m, since an odd base always keeps every one of its powers odd.

3. Prove directly: "If $x, y > 0$ and $x^2 > y^2$, then $x > y$."

Answer: Proved.

Method: Since $x^2 > y^2$, we have $x^2 - y^2 > 0$, which factors as $(x - y)(x + y) > 0$. Since $x, y > 0$, $x + y > 0$. Dividing the inequality by the positive quantity $x + y$ preserves its direction: $x - y > 0$, i.e. $x > y$.

> **Investigate further:** This is the rigorous justification behind "you can safely take square roots of an inequality between two positive quantities" — the positivity of $x + y$ is exactly what makes the division step safe.

4. Prove directly: "If n is a multiple of 3 but not a multiple of 9, then n^2 is a multiple of 9 but not a multiple of 81."

Answer: Proved.

Method: Write $n = 3m$ where $3 \nmid m$ (since n is a multiple of 3 but not $9 = 3 \times 3$, the quotient $m = n/3$ cannot itself be a multiple of 3). Then $n^2 = 9m^2$, manifestly a multiple of 9. Is n^2 a multiple of $81 = 9^2$? That would require m^2 to be a multiple of 9, which (since 3 is prime) would force $3 \mid m$ — contradicting $3 \nmid m$. So m^2 is not a multiple of 9, and $n^2 = 9m^2$ is not a multiple of 81.

> **Investigate further:** This reuses the "3 prime $\Rightarrow (3 \mid m^2 \iff 3 \mid m)$" fact from B2/Day 1's C1 as a load-bearing step inside a longer argument — exactly the payoff of having proved it once cleanly.

5. Prove by contrapositive: "If x and y are real numbers with xy irrational, then $x \neq 0$ and $y \neq 0$."

Answer: Proved via the contrapositive "if $x = 0$ or $y = 0$, then xy is rational".

Method: The contrapositive of "$P \Rightarrow (Q$ and $R)$" is "(not Q or not $R) \Rightarrow$ not P", i.e. here "$x = 0$ or $y = 0 \Rightarrow xy$ is rational". If $x = 0$ (or symmetrically $y = 0$), then $xy = 0$, which is an integer, hence rational.

> **Investigate further:** This is a genuinely different shape of contrapositive from earlier in the sheet: the original conclusion was a compound "and" statement, so negating it correctly requires De Morgan (Day 1's toolkit) before the contrapositive can even be written down.

6. Prove by contrapositive: "If n is an integer such that $2n^2 + 1$ is a multiple of 3, then n is not a multiple of 3."

Answer: Proved via the contrapositive "if n is a multiple of 3, then $2n^2 + 1$ is not a multiple of 3".

Method: Let $n = 3k$. Then $2n^2 + 1 = 2(9k^2) + 1 = 18k^2 + 1$. Modulo 3: $18k^2 \equiv 0 \pmod 3$ (since $18 = 3 \times 6$), so $2n^2 + 1 \equiv 1 \pmod 3$, which is not a multiple of 3.

> **Investigate further:** This is the sheet's first contrapositive argument to work modulo 3 rather than modulo 2 — the same "substitute and reduce" mechanics, just with a different modulus, confirming the technique isn't tied to parity specifically.

7. Prove directly: "If n is a positive integer that is not a multiple of 3, then $n^2 \equiv 1 \pmod 3$."

Answer: Proved.

Method: Since n is not a multiple of 3, $n \equiv 1$ or $n \equiv 2 \pmod 3$. Case $n \equiv 1$: $n = 3k + 1$, so $n^2 = 9k^2 + 6k + 1 \equiv 1 \pmod 3$. Case $n \equiv 2$: $n = 3k + 2$, so $n^2 = 9k^2 + 12k + 4 \equiv 4 \equiv 1 \pmod 3$. Both cases give $n^2 \equiv 1 \pmod 3$.

> **Investigate further:** This is the sheet's first genuinely case-split direct proof — unlike the single-substitution proofs elsewhere in Sections A–C, "not a multiple of 3" splits into two residues that both need checking before the claim is established.

8. True or false, with justification: "A statement with no counterexamples found after checking 100 cases has therefore been proven true by direct proof."

Answer: False.

Method: Direct proof requires a general logical argument valid for every case in the statement's domain, not exhaustive case-checking (which is only a proof when the domain itself is finite and every case truly has been checked). Checking 100 cases of a statement about, say, all positive integers leaves infinitely many cases unchecked — the statement could fail at case 101 or far beyond.

> **Investigate further:** The famous historical example is Euler's conjecture that n^2+n+41 is always prime, which holds for $n=0,\dots,39$ (checking 40 cases with no counterexample) before failing at $n=40$ — a stark reminder that pattern-matching from small cases is never a substitute for a general argument.

Section D Challenge

1. A region is defined by the inequalities $x+y>4$ and $x-y>-2$. Consider the three statements: I. $x>1$ II. $y>2$ III. $(x+y)(x-y)>-12$. Which of the above statements is/are true for every point in the region? *(after TMUA 2016 Paper 2 Q8)*

Answer: B) I only.

Method: The boundary lines $x+y=4$ and $x-y=-2$ intersect where adding the equations gives $2x=2$, so $x=1, y=3$. Rewriting the constraints as $y>4-x$ and $y<x+2$ requires $4-x<x+2$, i.e. $x>1$ — so statement I holds for every point in the region (this is forced by the region's very existence at each x, not just checked at the corner). Statement II fails: take $x=10, y=-5$ (check: $x+y=5>4$ and $x-y=15>-2$, both satisfied), but $y=-5\not>2$. Statement III fails: take $x=49.05, y=50.95$ (so $x-y=-1.9>-2$ and $x+y=100>4$, both satisfied), giving $(x+y)(x-y)=100\times(-1.9)=-190\not>-12$.

> **Investigate further:** Statement I is provable directly from the region's defining inequalities (an algebraic fact true everywhere in the region); statements II and III instead need a specific constructed counterexample — the two Day-3 techniques (direct proof, and disproof via a found instance) side by side in one question.

2. Prove by contrapositive: "If n^2-2n is even, then n is even."

Answer: Proved via the contrapositive "if n is odd, then n^2-2n is odd".

Method: Let $n=2k+1$. Then $n^2-2n=n(n-2)=(2k+1)(2k-1)=4k^2-1=2(2k^2-1)+1$, one more than an even number, hence odd.

> **Investigate further:** Direct proof would require assuming n^2-2n is even and somehow extracting n's parity from the factored form $n(n-2)$ — messier than confirming the even case forces an even product, exactly Day 3's B3 lesson about when contrapositive is the natural choice.

3. Prove directly: "If $n>4$ is such that both $n-1$ and $n+1$ are prime, then n is a multiple of 6."

Answer: Proved.

Method: Since $n>4$, both $n-1>3$ and $n+1>5$ are primes greater than 3, hence odd (the only even prime is 2) — so n, sandwiched between two odd numbers, is even. Among the three consecutive integers $n-1, n, n+1$, exactly one is a multiple of 3; since $n-1$ and $n+1$ are primes greater than 3, neither is a multiple of 3, so n itself must be. Being both even and a multiple of 3, n is a multiple of 6.

> **Investigate further:** The condition $n>4$ is doing real work: at $n=4$, $n-1=3$ and $n+1=5$ are both prime, yet 4 is not a multiple of 6 — the argument's "both primes exceed 3" step genuinely requires $n>4$, not just $n>2$.

4. Prove directly: "If x and y are positive reals with $x\neq y$, then $\dfrac{x+y}{2}>\sqrt{xy}$."

> **Answer:** Proved.

Method: Since $x \neq y$, $\sqrt{x} \neq \sqrt{y}$, so $(\sqrt{x} - \sqrt{y})^2 > 0$ (a nonzero real squared is strictly positive). Expanding: $x - 2\sqrt{xy} + y > 0$, so $x + y > 2\sqrt{xy}$. Dividing by 2 gives $\frac{x+y}{2} > \sqrt{xy}$.

> **Investigate further:** This is the strict AM–GM inequality for two positive reals, proved by the same "complete the square, use that a nonzero square is positive" trick that will reappear constantly — the equality case $x = y$ is exactly what's excluded by the hypothesis.

5. Statement T: "If n is a positive integer such that n^2 is a multiple of 6, then n is a multiple of 6." (a) Is T true? Prove or disprove. (b) Contrast with the fact that "n^2 is a multiple of 4" does *not* force "n is a multiple of 4" (witnessed by $n = 6$). What property of 6, that 4 lacks, makes T true?

> **Answer:** (a) T is true. (b) $6 = 2 \times 3$ is squarefree (a product of distinct primes); $4 = 2^2$ is not.

Method: (a) n^2 a multiple of 6 means n^2 is a multiple of both 2 and 3. Since 2 is prime, $2 \mid n^2 \Rightarrow 2 \mid n$; since 3 is prime, $3 \mid n^2 \Rightarrow 3 \mid n$. So n is a multiple of both 2 and 3, and since $\gcd(2, 3) = 1$, n is a multiple of $\operatorname{lcm}(2, 3) = 6$. (b) For a prime p, $p \mid n^2 \Rightarrow p \mid n$ always (this is what makes each individual step in part (a) work). But for a prime *square* like $4 = 2^2$, $4 \mid n^2$ only forces $2 \mid n$ (one factor of 2 in n^2's exponent-doubling is enough to reach exponent 2), not $4 \mid n$ — exactly why $n = 6$ (even but not a multiple of 4) has $n^2 = 36$, a multiple of 4, without n being one.

> **Investigate further:** This closes Day 3 by explaining, not just observing, the recurring 6-versus-4 pattern that has appeared since Day 1 (C1 vs C8) — squarefree moduli transfer divisibility from n^2 back to n cleanly; prime-power moduli generally do not.