Daily Logic Drill #2 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Implication, Converse, Inverse & Contrapositive · Necessary vs Sufficient.

Section A Rapid Recognition

1. State the converse of: "If n is a multiple of 6, then n is a multiple of 3."

Answer: "If n is a multiple of 3, then n is a multiple of 6."

Method: Swap the hypothesis and conclusion: the converse of "if P then Q" is "if Q then P".

> **Investigate further:** This converse is false ($n = 3$ is a multiple of 3 but not 6) — Section B's B10 explores a nearly identical pair.

2. State the contrapositive of: "If n is a multiple of 6, then n is a multiple of 3."

Answer: "If n is not a multiple of 3, then n is not a multiple of 6."

Method: Negate both clauses and swap: the contrapositive of "if P then Q" is "if not Q then not P".

> **Investigate further:** Unlike A1's converse, the contrapositive is always true whenever the original is (A7) — since the original is true ($6 = 3 \times 2$), this contrapositive is true too.

3. True or false: "$x = 3$" is sufficient for "$x^2 = 9$".

Answer: True.

Method: $x = 3 \Rightarrow x^2 = 9$ directly by substitution — exactly what "sufficient" means.

> **Investigate further:** Compare A4: sufficiency and necessity are independent questions about the same pair of statements.

4. True or false: "$x = 3$" is necessary for "$x^2 = 9$".

Answer: False.

Method: $x^2 = 9 \Rightarrow x = 3$ is false: $x = -3$ also satisfies $x^2 = 9$.

> **Investigate further:** "$x = 3$" is sufficient but not necessary for "$x^2 = 9$" — the first fully classified pair of the day.

5. Is "n is a multiple of 4" necessary, sufficient, or neither, for "n is a multiple of 2"?

Answer: Sufficient.

Method: $4 \mid n \Rightarrow 2 \mid n$, since $4 = 2 \times 2$. It is not necessary: $n = 2$ is a multiple of 2 but not of 4.

> **Investigate further:** A quick way to spot sufficiency instantly: whenever the first property is the more restrictive one (multiples of 4 are a subset of multiples of 2), it can only ever be sufficient, never necessary.

6. State the inverse of: "If n is prime, then n is odd or $n = 2$."

Answer: "If n is not prime, then n is even and $n \neq 2$."

Method: The inverse negates both clauses without swapping their order: "if not P then not Q". Negating "odd or $n = 2$" gives, by De Morgan, "not odd and $n \neq 2$", i.e. "even and $n \neq 2$".

> **Investigate further:** The inverse is a distinct operation from both the converse (swap only) and the contrapositive (negate and swap) — Section B's B7 investigates how the inverse relates to the converse.

7. True or false: "The contrapositive of a true statement is always true."

Answer: True.

Method: "If P then Q" is false only when P is true and Q is false; "if not Q then not P" is false only in that same case. They share the same truth value in every case, so a true original always has a true contrapositive.

> **Investigate further:** This is the entire reason proof by contrapositive (a later toolkit item) is a fully legitimate proof method, not just a rephrasing trick.

8. True or false: "The converse of a true statement is always true."

Answer: False.

Method: A1 is a direct counterexample: "n mult. $6 \Rightarrow n$ mult. 3" is true, but its converse is false.

> **Investigate further:** Contrast directly with A7 — the contrapositive is always safe, the converse never is guaranteed.

9. Negate: "If n is even, then n^2 is even."

Answer: "n is even and n^2 is odd."

Method: The negation of "if P then Q" is "P and not Q", never another conditional.

> **Investigate further:** This negation is unsatisfiable (no even n gives an odd n^2), consistent with the original being true.

10. True or false: "n is a multiple of 12" is sufficient for "n is a multiple of 4".

Answer: True.

Method: $12 = 4 \times 3$, so $12 \mid n \Rightarrow 4 \mid n$ directly.

> **Investigate further:** Not necessary: $n = 4$ is a multiple of 4 but not of 12 — the same "more restrictive $\Rightarrow$ sufficient only" pattern as A5.

Section B Manipulation Drills

1. State both the converse and contrapositive of "If $x > 2$, then $x^2 > 4$." Determine the truth of each.

Answer: Converse: "If $x^2 > 4$, then $x > 2$" — false. Contrapositive: "If $x^2 \leq 4$, then $x \leq 2$" — true.

Method: Converse counterexample: $x = -3$ gives $x^2 = 9 > 4$ but $x \leq 2$. The contrapositive shares its truth value with the (true) original: $x^2 \leq 4 \iff -2 \leq x \leq 2$, which certainly forces $x \leq 2$.

> **Investigate further:** A false converse alongside a true contrapositive is the default pattern, not a coincidence — only the contrapositive is guaranteed to inherit the original's truth.

2. Classify "$n = 6$" as necessary, sufficient, both, or neither, for "n is a multiple of 3".

Answer: Sufficient but not necessary.

Method: $n = 6 \Rightarrow n$ is a multiple of 3 (sufficient). $n = 9$ is a multiple of 3 but $n \neq 6$ (not necessary).

> **Investigate further:** A single specific value can be sufficient for a broader property, but is essentially never necessary for one — necessity requires ruling out every other possibility.

3. Classify "n is divisible by 6" as necessary, sufficient, both, or neither, for "n is divisible by 2 and by 3".

Answer: Necessary and sufficient.

Method: Sufficient: $6 \mid n \Rightarrow 2 \mid n$ and $3 \mid n$ directly. Necessary: since $\gcd(2, 3) = 1$, $\mathrm{lcm}(2, 3) = 6$, so $2 \mid n$ and $3 \mid n$ together force $6 \mid n$.

> **Investigate further:** Whenever a statement and its converse are both true, the condition is necessary and sufficient — Section B's B5 makes this equivalence explicit.

4. State the contrapositive of "If a and b are both even, then $a + b$ is even." Is it true?

Answer: "If $a + b$ is odd, then a is odd or b is odd." True.

Method: Negate both clauses and swap: hypothesis becomes "$a + b$ is odd", conclusion becomes "a is odd or b is odd" (De Morgan on "not(both even)"). It shares the original's truth value, and the original is true: $a = 2j, b = 2k \Rightarrow a + b = 2(j + k)$, even.

> **Investigate further:** Writing the contrapositive of a compound hypothesis ("both even") forces a De Morgan step on the way — exactly Day 1's negation toolkit, now applied to a conditional's clauses.

5. True or false: "If $P \Rightarrow Q$ and $Q \Rightarrow P$ are both true, then P and Q are necessary and sufficient for each other."

Answer: True.

Method: $P \Rightarrow Q$ is exactly "P is sufficient for Q"; $Q \Rightarrow P$ is exactly "P is necessary for Q" (since it says Q requires P). Having both gives necessary and sufficient by definition.

> **Investigate further:** This is B3's underlying principle stated in full generality — "iff" is nothing more than a statement and its converse both holding.

6. Given that "x is rational" is sufficient for "x^2 is rational", state the converse. Is the converse true?

Answer: Converse: "x^2 is rational" is sufficient for "x is rational". The converse is false.

Method: Counterexample: $x = \sqrt{2}$ is irrational, but $x^2 = 2$ is rational — so x^2 rational does not force x rational.

> **Investigate further:** Squaring can destroy irrationality even though multiplying two rationals never introduces it — the same asymmetry underlies why $\sqrt{2}$'s irrationality proof works at all.

7. State the inverse and the converse of "If n is prime, then n is odd or $n = 2$." Are they logically equivalent to each other?

Answer: Inverse: "If n is not prime, then n is even and $n \neq 2$" (A6). Converse: "If n is odd or $n = 2$, then n is prime." Yes, they are logically equivalent.

Method: In general, the contrapositive of the converse "$Q \Rightarrow P$" is "not $P \Rightarrow$ not Q", which is exactly the inverse. Since a statement and its contrapositive always share a truth value (A7), the converse and the inverse — being contrapositives of each other — must always share a truth value too.

> **Investigate further:** This is a genuinely useful shortcut: to find whether a converse is true, it is sometimes easier to test the inverse instead, since they are guaranteed to agree.

8. True or false: "$a = b$" is necessary and sufficient for "$a^2 = b^2$".

Answer: False.

Method: Sufficient: $a = b \Rightarrow a^2 = b^2$, true. Necessary: $a^2 = b^2 \Rightarrow a = b$? False — $a = 2, b = -2$ gives $a^2 = b^2 = 4$ but $a \neq b$. So it is sufficient but not necessary, not both.

> **Investigate further:** The correct necessary-and-sufficient condition is $a = b$ *or* $a = -b$ — squaring always loses sign information (Day 1's D5 previewed this exact pattern with a function).

9. Is "$x^2 > 4$" necessary, sufficient, both, or neither, for "$x > 2$"?

Answer: Necessary but not sufficient.

Method: Necessary: $x > 2 \Rightarrow x^2 > 4$, true. Not sufficient: $x = -3$ gives $x^2 = 9 > 4$ but $x \not> 2$.

> **Investigate further:** $x^2 > 4 \iff x > 2$ or $x < -2$ — the "$x < -2$" branch is exactly what breaks sufficiency here.

10. State the converse of "If n is a multiple of 9, then n is a multiple of 3." Is the converse true?

Answer: Converse: "If n is a multiple of 3, then n is a multiple of 9." The converse is false.

Method: Counterexample: $n = 3$ is a multiple of 3 but not of 9.

> **Investigate further:** This is the exact converse-failure pair used again in Section D's D5(b), where it's contrasted with the (true) 12-vs-4-and-3 relationship.

Section C Substitution & Structure

1. For real numbers x, y, the condition "$x + y$ is an integer and $x - y$ is an integer" is:

Answer: A) necessary but not sufficient for "x and y are both integers".

Method: Necessary: if x, y are both integers, then $x + y$ and $x - y$ are automatically both integers. Not sufficient: $x = y = 0.5$ gives $x + y = 1$ and $x - y = 0$, both integers, but x, y themselves are not — ruling out options B and C. Necessity alone rules out D.

> **Investigate further:** Solving $x = \frac{(x+y)+(x-y)}{2}$, $y = \frac{(x+y)-(x-y)}{2}$ shows exactly what goes wrong: x, y are only forced to be integers when $x+y$ and $x-y$ have the *same parity* (so the sum/difference is even) — a condition the question's hypothesis never guarantees.

2. Given that P is necessary and sufficient for Q, and Q is necessary and sufficient for R, prove that P is necessary and sufficient for R.

Answer: Proved.

Method: "P necessary and sufficient for Q" gives both $P \Rightarrow Q$ and $Q \Rightarrow P$; likewise $Q \Rightarrow R$ and $R \Rightarrow Q$. Chaining sufficiency: $P \Rightarrow Q \Rightarrow R$ gives $P \Rightarrow R$. Chaining the reverse: $R \Rightarrow Q \Rightarrow P$ gives $R \Rightarrow P$. Both directions hold, so P is necessary and sufficient for R.

> **Investigate further:** "Iff" chains exactly like plain implication does, just in both directions at once — a chain of equivalent definitions transfers all the way down the chain.

3. Classify "a triangle has all three altitudes of equal length" as a condition for "the triangle is equilateral". Justify using the fact that an altitude to side a has length $\frac{2\times\text{Area}}{a}$.

Answer: C) necessary and sufficient.

Method: Let the altitudes to sides a, b, c be h_a, h_b, h_c, and let the triangle's area be A. Sufficient: if the triangle is equilateral, $a = b = c$, so $h_a = \frac{2A}{a} = \frac{2A}{b} = \frac{2A}{c} = h_b = h_c$ directly. Necessary: if $h_a = h_b = h_c$, then $\frac{2A}{a} = \frac{2A}{b} = \frac{2A}{c}$, and since $A \neq 0$, this forces $a = b = c$ — the triangle is equilateral.

> **Investigate further:** Both directions rely on the same identity ($h_a = \frac{2A}{a}$) read in opposite directions — side-length equality and altitude-length equality are simply the same fact viewed through a common factor of $2A$.

4. For positive integers a, b, n, suppose you're told only that n divides a, or n divides b (possibly both). Does this pin down whether n divides the product ab? And working the other way: does knowing n divides ab pin down whether n divides a or n divides b? *(after TMUA 2019 Paper 2 Q5)*

Answer: B) "n divides a or n divides b" is sufficient but not necessary for "n divides ab".

Method: Sufficient: if $n \mid a$ (or $n \mid b$), then n divides the product ab directly. Not necessary: taking $n = 9$, $a = 3$, $b = 3$ gives $ab = 9$, divisible by 9, but neither a nor b individually is divisible by 9 — ruling out options A and C. Option D is ruled out by the sufficiency direction alone.

> **Investigate further:** This only fails because $n = 9 = 3^2$ is not squarefree — if n were prime, divisibility of a product would force divisibility of a factor (Euclid's Lemma), making the condition necessary too. The gap here is exactly why primality matters so much in Number Theory.

5. Statement S: "If n is a multiple of 8, then n is a multiple of 2." Determine whether S's converse is true by testing S's inverse instead. State the inverse, and your conclusion about the converse.

Answer: Inverse: "If n is not a multiple of 8, then n is not a multiple of 2." The inverse is false, so the converse is also false.

Method: $n = 4$ is not a multiple of 8, but 4 is a multiple of 2 — a direct counterexample to the inverse. By B7's fact (inverse $\equiv$ converse always), the converse must therefore be false too, without needing to test it separately: the converse "if n is a multiple of 2, then n is a multiple of 8" indeed fails at $n = 2$.

> **Investigate further:** This is B7's shortcut actually put to work: one counterexample check (on the inverse) settled the converse's truth value for free.

6. Classify "n^2 is even" as a condition for "n is even".

Answer: C) necessary and sufficient.

Method: Sufficient: if n^2 is even, then n cannot be odd (an odd number squared is odd), so n is even. Necessary: if n is even, n^2 is even directly.

> **Investigate further:** Both directions holding here is exactly why "n even $\iff$ n^2 even" is such a reliable tool — contrast with C5's example of an inverse that is *false*, which drags its converse down with it.

7. Statement S is true, and S's inverse is also true. What can you conclude about S's converse? Justify.

Answer: S's converse must also be true.

Method: By B7, S's inverse and S's converse are always logically equivalent (same truth value in every case). Since the inverse is given true, the converse is true too.

> **Investigate further:** This gives a free extra fact: checking the inverse instead of the converse directly can sometimes be the easier route to the same conclusion.

8. State the contrapositive of "If a and b are both irrational, then ab is irrational." Is the original statement true? Give a counterexample if false.

Answer: Contrapositive: "If ab is rational, then a is rational or b is rational." The original statement is false; counterexample $a = b = \sqrt{2}$.

Method: $\sqrt{2}$ is irrational, but $ab = \sqrt{2} \times \sqrt{2} = 2$, which is rational — disproving the original. Since the contrapositive always shares the original's truth value (A7), the same counterexample shows the contrapositive is false too: $ab = 2$ is rational, yet neither $a = \sqrt{2}$ nor $b = \sqrt{2}$ is rational.

> **Investigate further:** One counterexample disproves both a statement and its contrapositive simultaneously — there is never a need to separately hunt for a second one.

Section D Challenge

1. Let $f(x) = x^2 + bx + c$ with $f(0) = c$. Let P be the statement: "if $c < 0$, then f has two distinct real roots." Which of the following must be true? *(after TMUA 2022 Paper 2 Q5)*

Answer: F) I and III only.

Method: I (P): the discriminant is $b^2 - 4c$; if $c < 0$ then $-4c > 0$, so $b^2 - 4c > b^2 \geq 0$, meaning the discriminant is always strictly positive — f always has two distinct real roots. P is true. II (converse): "if f has two distinct real roots, then $c < 0$" — false, since $b = 3, c = 1$ gives discriminant $9 - 4 = 5 > 0$ (two distinct real roots) with $c = 1 \not< 0$. III (contrapositive): shares P's truth value (A7), so it is true.

> **Investigate further:** This mirrors B1's pattern exactly (true original, false converse, true contrapositive) but now requires an actual discriminant computation to establish P itself, rather than reading it off directly.

2. Consider a finite list of n real numbers (repeats allowed), sorted into non-decreasing order. Let

P be the statement "n is odd", and let Q be the statement "the middle value of the sorted list equals one of the list's own entries". P is: *(after TMUA 2022 Paper 2 Q6)*

> **Answer:** B) sufficient but not necessary for Q.

Method: Sufficient: if n is odd, the list has a unique middle term when sorted, and that middle term *is* the median — a list member. Not necessary: the list $4, 4$ has $n = 2$ (even), and the median is $\frac{4+4}{2} = 4$, which is a list member — so Q holds even though P fails, ruling out options A and C. Sufficiency alone rules out D.

> **Investigate further:** Any even-length list whose two middle entries are equal gives the same kind of counterexample — it is n being odd that *guarantees* Q, not the only way to achieve it.

3. For a positive integer n, let P be "n leaves remainder 1 when divided by 4" and Q be "n^2 leaves remainder 1 when divided by 8." P is:

> **Answer:** B) sufficient but not necessary for Q.

Method: Sufficient: if $n = 4k + 1$, then $n^2 = 16k^2 + 8k + 1$, and modulo 8 this is 1 (both $16k^2$ and $8k$ are multiples of 8). Not necessary: if $n = 4k + 3$, then $n^2 = 16k^2 + 24k + 9 = 16k^2 + 24k + 8 + 1$, and modulo 8 this is also 1 (all of $16k^2, 24k, 8$ are multiples of 8) — so Q can hold with $n \equiv 3 \pmod 4$, where P fails, ruling out A and C. Sufficiency alone rules out D.

> **Investigate further:** The true necessary-and-sufficient condition for Q is simply "n is odd" — every odd square is $\equiv 1 \pmod 8$, regardless of which odd residue mod 4 it comes from.

4. Statement S: "If a quadrilateral has four equal sides, then its diagonals are perpendicular." (a) Is S true? (b) State S's converse. Is the converse true?

> **Answer:** (a) True (a rhombus). (b) Converse: "If a quadrilateral's diagonals are perpendicular, then it has four equal sides." The converse is false.

Method: (a) Let quadrilateral $WXYZ$ have $WX = XY = YZ = ZW$, and let M be the intersection of diagonals WY and XZ. Triangles WXY and WZY are congruent (SSS: $WX = WZ$, $XY = ZY$, WY shared), so $\angle XWY = \angle ZWY$, meaning WY bisects $\angle XWZ$; since triangle WXZ is isosceles ($WX = WZ$), this bisector is also the perpendicular bisector of XZ. So the diagonals meet at right angles. (b) A kite with $WX = WZ$ but $XY = ZY \neq WX$ (two pairs of adjacent equal sides, not all four equal) has the same SSS congruence argument still apply to triangles WXY, WZY — so its diagonals are still perpendicular, but it is a counterexample to the converse since its sides are not all equal.

> **Investigate further:** "Four equal sides" is a strictly stronger condition than "diagonals perpendicular" — every rhombus is a kite, but not every kite is a rhombus, matching the sufficient-not-necessary pattern threaded through the whole sheet, now geometric.

5. Statement S: "If n is a positive integer such that $n^2 - 1$ is prime, then $n = 2$." (a) Is S true? Prove or disprove. (b) State S's contrapositive. (c) Is "$n^2 - 1$ is prime" necessary and sufficient for "$n = 2$"?

> **Answer:** (a) True. (b) "If $n \neq 2$, then $n^2 - 1$ is not prime." (c) Yes.

Method: (a) Factor $n^2 - 1 = (n - 1)(n + 1)$. For a product of two positive integers to be prime, one factor must equal 1 (a prime has exactly two positive divisors, itself and 1, so the smaller factor of any factorisation into two positive integers > 1 would already give at least two nontrivial divisors). Since $n \geq 1$, the smaller factor is $n - 1$, forcing $n - 1 = 1$, i.e. $n = 2$; checking, $n = 2$ gives $n^2 - 1 = 3$, which is indeed prime. For every other $n > 2$, both $n - 1 \geq 2$ and $n + 1 \geq 4$, so $(n - 1)(n + 1)$ is a product of two integers each > 1, hence composite. So $n = 2$ is true, and it is the *only* positive integer for which $n^2 - 1$ is prime. (b) Negate both clauses of S and swap. (c) Since $n = 2$ is shown above to be the unique solution, "$n^2 - 1$ prime" and "$n = 2$" describe exactly the same set of positive integers — necessary and sufficient.

> **Investigate further:** This closes the day by using all four of its tools in sequence on a single fact: stating S (Day 2's headline object), proving it directly via factoring, writing its contrapositive on demand, and finally recognising that a "prove S" result about a *unique* solution automatically upgrades to a full necessary-and-sufficient classification — uniqueness is exactly what turns a one-directional implication into an iff for free.