Daily Combinatorics Drill #7

Sheet by David Akinyele-Aje

Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Recurrences & Fibonacci Counting · Catalan Numbers · Bijections · Invariants & Monovariants.

Attempt each question before checking answers. Time yourself. No calculators.

Section A Rapid Recognition (≤15 s each)

Small recurrence and bijection values on sight. Build up, don't brute-force.

1. In how many ways can you climb 4 stairs, taking 1 or 2 steps at a time?

2. And 5 stairs?

3. How many binary strings of length 4 have no two adjacent 1s?

4. In how many ways can a 2×3 rectangle be tiled by dominoes?

5. And a 2×4 rectangle?

6. Into how many regions do 3 lines in general position divide the plane?

7. In how many ways can 3 pairs of brackets be written so that they match correctly?

8. A sequence has $a_1 = 1$ and $a_{n+1} = 2a_n + 1$. Find a_4.

9. The Fibonacci numbers are $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. Write down F_8.

10. True or false: $F_1 + F_2 + \cdots + F_8 = 54$.

Section B Manipulation Drills (≤50 s each)

Set up the recurrence, then run it. Name the state before you count.

1. In how many ways can you climb 10 stairs, taking 1 or 2 steps at a time?

2. How many binary strings of length 8 have no two adjacent 1s?

3. In how many ways can a 2×10 rectangle be tiled by dominoes?

4. Into how many regions do 10 lines in general position divide the plane?

5. A sequence has $a_1 = 2$ and $a_n = 3a_{n-1} - 1$. Find a_4.

6. In how many ways can you climb 6 stairs taking 1, 2, or 3 steps at a time?

7. Five points lie on a circle and every pair is joined by a chord, with no three chords concurrent inside. Into how many regions is the disc divided?

8. A 4×6 chocolate bar is snapped along its gridlines into 24 unit squares, one snap splitting one piece into two (no stacking). What is the smallest number of snaps needed?

9. What is the least number of moves needed to solve the Towers of Hanoi with 6 disks?

10. Given the identity $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$, evaluate $F_4 + F_5 + \cdots + F_{12}$.

Section C Substitution & Structure (≤90 s each)

Two-state recurrences, tilings and bijections. Find the object being counted twice.

1. In how many ways can you climb 8 stairs, taking 1 or 2 steps, with no two 2-steps in a row?

2. In how many ways can a 2×6 strip be tiled using 1×2 dominoes and 2×2 squares?

3. How many lattice paths run from $(0, 0)$ to $(4, 4)$ using unit right and up steps and never rising above the line $y = x$?

4. Six points lie on a circle; every pair is joined by a chord, no three concurrent inside. Into how many regions is the disc divided? *(the answer is not* 32*)*

5. In how many ways can 8 be written as an ordered sum of odd positive integers?

6. How many strings of length 6 over the alphabet $\{0, 1, 2\}$ contain no two adjacent 0s?

7. In how many ways can a 3×4 grid be tiled by dominoes?

8. How many binary strings of length 10 contain no three consecutive 1s?

Section D Challenge (BMO1 difficulty)

Determine, with proof. The write-up is the answer — state the bijection or invariant in full.

1. Let a_n be the number of permutations p of $\{1, 2, \ldots, n\}$ with $|p(i) - i| \leq 1$ for every i. Prove that $a_n = a_{n-1} + a_{n-2}$, and hence find a_6.

2. The numbers $1, 2, \ldots, 10$ are written on a board. Repeatedly, two numbers are erased and their (non-negative) difference is written in their place, until one number remains. Prove that the final number is odd.

3. Prove that the number of ways to triangulate a convex hexagon by non-crossing diagonals is 14.

4. Prove that any $2^n \times 2^n$ board with one unit square removed can be tiled by L-shaped trominoes.

5. Two players alternately remove a number of stones equal to some power of 2 (that is, $1, 2, 4, 8, \ldots$) from a pile of 2025, and the player who takes the last stone wins. Determine, with proof, which player has a winning strategy.

"A recurrence remembers where it came from; an invariant refuses to forget. Between them they count almost everything."

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths