Daily Combinatorics Drill #6

Sheet by David Akinyele-Aje

Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Pigeonhole Principle (basic & strong) · Worst-Case Thinking · Double Counting · The Handshake Lemma.

Attempt each question before checking answers. Time yourself. No calculators.

Section A Rapid Recognition (≤15 s each)

Guarantees on sight. Ask: what is the worst case?

1. A drawer holds 10 black and 10 white socks. How many must be drawn (in the dark) to guarantee a matching pair?
2. Same drawer: how many draws guarantee a *black* pair?
3. True or false: among any 13 people, two share a birth month.
4. How many people are needed to guarantee two were born on the same day of the week?
5. How many integers must you take to guarantee two of the same parity?
6. Six friends each count their handshakes at a party; the counts total 18. How many handshakes happened?
7. True or false: in any grid of numbers, the sum of the row totals equals the sum of the column totals.
8. How many integers must you take to guarantee two whose difference is a multiple of 5?
9. A bag has sweets of 4 flavours. How many sweets guarantee 3 of one flavour?
10. True or false: among any 4 integers, some two have an even difference.

Section B Manipulation Drills (≤50 s each)

Name the pigeonholes, or name what is being counted twice.

1. A round table has 12 seats. What is the smallest number of diners that forces two of them to sit in adjacent seats? *(after TMUA 2019 P2 Q9)*
2. How many integers must you take to guarantee two whose difference is divisible by 7?

3. Show that among any 5 integers, two have a difference divisible by 4. Then write down 5 integers of which no two differ by a multiple of *5*.

4. Six numbers are chosen from $\{1, 2, \ldots, 10\}$. Must two of them sum to 11? What is the smallest choice size that forces this? *(after TMUA 2022 P2 Q8)*

5. Eight teams play a round-robin (each pair meets once). How many matches are played, and how many does each team play?

6. In a class of 30, each student writes down their number of friends within the class, and the numbers total 84. How many friendships are there?

7. Four numbers are chosen from $\{1, 2, \ldots, 6\}$. Show that two of them must be consecutive, and exhibit a largest possible choice with no two consecutive.

8. At any party with at least two people, two attendees have shaken the same number of hands. What makes the counts 0 and $n - 1$ incompatible?

9. Twenty-five sweets are shared among 7 children. Show that some child receives at least 4.

10. A society has 5 committees, each with 4 members, and every member sits on exactly 2 committees. How many members does the society have?

Section C Substitution & Structure (≤90 s each)

Strong pigeonhole and structural double counts. Build the boxes before you count the pigeons.

1. A jar holds 10 red, 10 green and 10 blue beads. How many must be drawn (in the dark) to guarantee 5 of *some* colour? How many to guarantee 5 *red*?

2. Seven distinct numbers are chosen from $\{1, 2, \ldots, 12\}$. Show that two of them sum to 13, and show that six numbers are not enough to force this.

3. How many integers must you take to guarantee two ending in the same digit?

4. An 8×8 chessboard has two diagonally opposite corner squares removed. Can the remaining 62 squares be tiled by 31 dominoes? Justify your answer.

5. How large must a group be to guarantee that two members share both a birth month *and* a birth day-of-the-week?

6. A school runs 20 clubs, each with exactly 6 students, and every student belongs to exactly 3 clubs. How many students does the school have?

7. The numbers $1, 2, \ldots, 9$ are arranged around a circle in any order. Show that some three adjacent numbers have sum at least 15.

8. Five distinct numbers are chosen from $\{1, 2, \ldots, 8\}$. Show that two of them differ by exactly 4.

Section D Challenge (SMC / BMO1 difficulty)

Determine, with proof. Model your write-up on the answer file's — the argument is the answer.

1. Five points lie in a unit square. Prove that two of them are at distance at most $\frac{\sqrt{2}}{2}$.

2. Five points have integer coordinates. Prove that the midpoint of some pair of them also has integer coordinates.

3. Prove that at any party, the number of people who have shaken hands an odd number of times is even.

4. Twenty students each solve exactly two of the five problems on a quiz. Prove that some two students solved exactly the same pair of problems — indeed that some pair of problems was chosen by at least two students.

5. Among any six people, any two of whom are either friends or strangers, prove that there exist three mutual friends or three mutual strangers.

"The pigeonhole principle proves existence without ever finding the thing — mathematics at its most confident."

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths