Daily Combinatorics Drill #7 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Recurrences & Fibonacci Counting · Catalan Numbers · Bijections · Invariants & Monovariants.

Section A Rapid Recognition (≤15 s each)

1. Climb 4 stairs, steps of 1 or 2?

Answer: 5

Method: Ways to reach step n: $f(n) = f(n-1) + f(n-2)$ (last move was a 1 or a 2). With $f(1) = 1$, $f(2) = 2$: $1, 2, 3, 5$.

> **Investigate further:** This is Fibonacci in disguise — $f(n) = F_{n+1}$. Nearly every question on this sheet is one recurrence wearing a costume; learn to hear "the last move was X or Y" as "$f(n) = f(n-X) + f(n-Y)$".

2. Climb 5 stairs?

Answer: 8

Method: Continue the sequence one step: $f(5) = f(4) + f(3) = 5 + 3 = 8$.

> **Investigate further:** Extend the reflex to steps of 1 or 3: the recurrence becomes $f(n) = f(n-1) + f(n-3)$ — the Padovan/Narayana pattern. Write its first eight values.

3. Binary strings of length 4, no two adjacent 1s?

Answer: 8

Method: Let $g(n)$ count them. A valid string ends in 0 (then $g(n-1)$ ways) or in 01 (then $g(n-2)$): $g(n) = g(n-1) + g(n-2)$, with $g(1) = 2$, $g(2) = 3$. So $2, 3, 5, 8$.

> **Investigate further:** Same recurrence as the stairs — and as sheet 2's no-two-consecutive subsets (55 for $n = 8$). Three problems, one Fibonacci. Find the bijection between length-n binary strings with no "11" and stair-climbs of n+1.

4. Domino tilings of 2×3?

Answer: 3

Method: Tilings of $2 \times n$: the last column is one vertical domino ($t(n-1)$) or two horizontals filling two columns ($t(n-2)$): $t(n) = t(n-1) + t(n-2)$, $t(1) = 1$, $t(2) = 2$. So $1, 2, 3$.

> **Investigate further:** Fibonacci again ($t(n) = F_{n+1}$). Why do stairs, no-11 strings and $2 \times n$ tilings all obey it? Each has a "binary choice at the end that removes 1 or 2 units" — name that choice in

each and the shared structure becomes visible.

5. Domino tilings of 2×4?

Answer: 5

Method: $t(4) = t(3) + t(2) = 3 + 2 = 5$.

Investigate further: Predict 2×10 before doing B3 (it is F_{11}). Fluency at extending a known recurrence in your head is the entire skill this section trains.

6. Regions from 3 lines in general position?

Answer: 7

Method: Each new line crosses all previous ones and gains (number of lines already present)+1 new regions: $R(n) = R(n-1) + n$, $R(0) = 1$. So $1, 2, 4, 7$.

Investigate further: Unfolding the recurrence gives $R(n) = 1 + \binom{n+1}{2}$ — a closed form from a running sum. Check $R(3) = 1 + 6 = 7$. What if lines may be parallel or concurrent? (Fewer regions — generality is what makes the count maximal.)

7. Correctly matched arrangements of 3 bracket pairs?

Answer: 5

Method: The Catalan number $C_3 = 5$: `((()))`, `(()())`, `(())()`, `()(())`, `()()()`.

Investigate further: $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts balanced brackets, and also triangulations (D3), staircase lattice paths (C3), and much else. The recurrence $C_n = \sum_i C_i C_{n-1-i}$ "splits at the matching partner of the first bracket" — meet it properly in C3.

8. $a_1 = 1$, $a_{n+1} = 2a_n + 1$: find a_4.

Answer: 15

Method: $1 \to 3 \to 7 \to 15$. (Each term is one less than a power of 2.)

Investigate further: Guess and prove the closed form $a_n = 2^n - 1$ by induction. This exact recurrence is the Towers of Hanoi (B9, C6) — recognising it now will save you the derivation later.

9. Write F_8.

Answer: 21

Method: $1, 1, 2, 3, 5, 8, 13, 21$.

Investigate further: Memorise F_1 through F_{12} $(\ldots, 34, 55, 89, 144)$ — they surface constantly. Note $F_{12} = 144 = 12^2$, the only nontrivial square Fibonacci number (a deep theorem).

10. True or false: $F_1 + \cdots + F_8 = 54$.

Answer: True

Method: $1{+}1{+}2{+}3{+}5{+}8{+}13{+}21 = 54$. Also $F_{10} - 1 = 55 - 1 = 54$.

Investigate further: The shortcut $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ is B10 — prove it and you never add

> Fibonacci numbers by hand again. Which running-sum identities does Pascal's triangle hide? (Hockey stick, sheet 4 D4.)

Section B Manipulation Drills

(≤ 50 s each)

1. Climb 10 stairs, steps of 1 or 2?

Answer: 89

Method: Run $f(n) = f(n-1) + f(n-2)$ to $n = 10$: $\ldots, 55, 89$. That is F_{11}.

> **Investigate further:** At $n = 10$ brute force would list 89 sequences; the recurrence gives it in ten additions. This gap widens fast — $f(30) = F_{31} = 1\,346\,269$. Recurrences are how counting scales.

2. Binary strings of length 8, no two adjacent 1s?

Answer: 55

Method: $g(n) = g(n-1) + g(n-2)$, $g(1) = 2, g(2) = 3$: $\ldots, 34, 55$. ($g(n) = F_{n+2}$.)

> **Investigate further:** Sheet 2 D2 counted no-two-consecutive *subsets* of $\{1, \ldots, 8\}$ and also got 55 — because a subset is exactly a binary string (element in/out). Same object, two sheets apart; that is the pillar closing a loop.

3. Domino tilings of 2×10?

Answer: 89

Method: $t(n) = F_{n+1}$, so $t(10) = F_{11} = 89$.

> **Investigate further:** Stairs of 10 and 2×10 tilings both give 89 — exhibit a direct bijection (a vertical domino $\leftrightarrow$ a 1-step, a horizontal pair $\leftrightarrow$ a 2-step). A bijection is a *proof* the counts are equal, needing no formula.

4. Regions from 10 lines in general position?

Answer: 56

Method: $R(n) = 1 + \binom{n+1}{2} = 1 + \binom{11}{2} = 1 + 55 = 56$.

> **Investigate further:** Verify against the recurrence: $R(10) = R(9) + 10$. Closed form and recurrence must always agree — disagreement means an arithmetic slip in one, a free error-check.

5. $a_1 = 2$, $a_n = 3a_{n-1} - 1$: find a_4.

Answer: 41

Method: $2 \to 5 \to 14 \to 41$.

> **Investigate further:** Solve it in closed form: fixed point $a^* = \frac{1}{2}$, so $a_n - \frac{1}{2} = 3(a_{n-1} - \frac{1}{2})$, giving $a_n = \frac{1}{2} + \frac{3}{2} \cdot 3^{n-1}$. Check $a_4 = \frac{1}{2} + \frac{3}{2} \cdot 27 = 41$. The "subtract the fixed point" trick linearises any $a_n = ra_{n-1} + c$.

6. Climb 6 stairs, steps of 1, 2, or 3?

Answer: 24

Method: Tribonacci: $f(n) = f(n-1) + f(n-2) + f(n-3)$, $f(1) = 1, f(2) = 2, f(3) = 4$. So $1, 2, 4, 7, 13, 24$.

> **Investigate further:** Three step-sizes give three seed values and a three-term recurrence — the pattern generalises to any step set. What step sizes would make f satisfy $f(n) = f(n-1) + f(n-4)$? (Steps of 1 and 4.)

7. Regions of a disc from 5 points, all chords?

Answer: 16

Method: Regions $= 1 + \binom{n}{2} + \binom{n}{4}$: with $n = 5$, $1 + 10 + 5 = 16$.

> **Investigate further:** The values $1, 2, 4, 8, 16$ for $n = 1, \ldots, 5$ scream "2^{n-1}" — and then $n = 6$ gives 31, not 32 (C4). This is the most famous false pattern in mathematics; never trust a doubling sequence without proof.

8. Snaps to break a 4×6 bar into 24 squares (no stacking)?

Answer: 23

Method: Invariant: each snap turns one piece into two, so the number of pieces rises by exactly 1 per snap. Starting at 1 piece and ending at 24 needs $24 - 1 = 23$ snaps — *regardless* of the order or pattern of breaks.

> **Investigate further:** Your first monovariant: a quantity (piece count) that changes by a fixed amount every move, so the total number of moves is forced. The answer never depends on strategy — the mark of an invariant argument. How many snaps for an $m \times n$ bar? ($mn - 1$.)

9. Least moves, Towers of Hanoi, 6 disks?

Answer: 63

Method: $H(n) = 2H(n-1) + 1$ (move $n-1$ disks off, move the big one, move them back), $H(1) = 1$: $1, 3, 7, 15, 31, 63$. So $H(n) = 2^n - 1$.

> **Investigate further:** This is A8's recurrence exactly. Prove $2^n - 1$ is also a *lower* bound (the biggest disk must move, and the other $n-1$ must be stacked off it first and on it after) — optimality needs both a construction and a bound, just like every guarantee on sheet 6.

10. Use $F_1 + \cdots + F_n = F_{n+2} - 1$ to evaluate $F_4 + \cdots + F_{12}$.

Answer: 372

Method: Subtract two prefix sums: $F_4 + \cdots + F_{12} = (F_1 + \cdots + F_{12}) - (F_1 + F_2 + F_3) = (F_{14} - 1) - (F_5 - 1) = F_{14} - F_5 = 377 - 5 = 372$.

> **Investigate further:** The identity comes from telescoping $F_k = F_{k+2} - F_{k+1}$, which collapses $\sum_{k=1}^{n} F_k = F_{n+2} - F_2 = F_{n+2} - 1$ — prove it in one line. Then derive $\sum F_i^2 = F_n F_{n+1}$ the same way (from $F_i^2 = F_i F_{i+1} - F_{i-1} F_i$): a gorgeous identity with a one-line proof.

Section C Substitution & Structure ($\leq$90 s each)

1. Climb 8 stairs, steps 1 or 2, no two 2-steps in a row?

Answer: 19

Method: Track the last step: let $b(n)$ end in a 1-step, $c(n)$ end in a 2-step. A 1-step may follow anything: $b(n) = b(n-1) + c(n-1)$. A 2-step may not follow a 2-step: $c(n) = b(n-2)$. Building up to $n = 8$ gives total $b(8) + c(8) = 19$.

> **Investigate further:** One forbidden adjacency forces a *two-state* recurrence — you must remember what the last move was. This "carry the state" idea is the workhorse behind C8 and countless BMO1 process problems. Recompute allowing no two *1*-steps in a row instead; is the answer the same by symmetry?

2. Tile a 2×6 strip with dominoes and 2×2 squares?

Answer: 43

Method: Fill left to right: a vertical domino advances 1 column (a_{n-1}); two horizontal dominoes or one 2×2 square advance 2 columns (2 ways, $2a_{n-2}$). So $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = a_1 = 1$: $1, 1, 3, 5, 11, 21, 43$.

> **Investigate further:** A "$+2a_{n-2}$" term means *two* ways to cross the two-column gap — the coefficient counts the local choices. This is the general shape of transfer-matrix tiling counts. Solve the recurrence in closed form (characteristic roots 2 and -1).

3. Lattice paths $(0, 0) \to (4, 4)$, unit steps, never above $y = x$?

Answer: $C_4 = 14$

Method: These are *Dyck paths*; they are counted by the Catalan number $C_4 = \frac{1}{5}\binom{8}{4} = \frac{70}{5} = 14$. (Reflection principle: all $\binom{8}{4} = 70$ paths minus the $\binom{8}{3} = 56$ that cross the diagonal.)

> **Investigate further:** The reflection trick — flip a bad path after its first illegal step to biject with unconstrained paths to a shifted endpoint — is one of the most reusable ideas in combinatorics. Balanced brackets (A7), non-crossing chords and triangulations (D3) are all secretly Dyck paths. Find the bracket $\leftrightarrow$ path bijection (up-step = "(").

4. Regions of a disc from 6 points, all chords? *(not 32)*

Answer: 31

Method: $1 + \binom{6}{2} + \binom{6}{4} = 1 + 15 + 15 = 31$. Each interior crossing (a $\binom{n}{4}$: one per 4 points, sheet 2 D5) adds one region beyond the $1 + \binom{n}{2}$ that chords alone would give.

> **Investigate further:** The sequence $1, 2, 4, 8, 16, \mathbf{31}$ is the textbook warning against pattern-matching. Derive the formula honestly via Euler's $V - E + F = 2$: count vertices (points + crossings), edges (chord arcs), and read off faces. Getting 31 from Euler is deeply satisfying.

5. Ordered sums of 8 into odd parts?

Answer: 21

Method: Let $d(n)$ count compositions of n into odd parts. Peeling the first part (odd) gives $d(n) = d(n-1) + d(n-3) + d(n-5) + \cdots$; simplifying, $d(n) = d(n-1) + d(n-2)$ — Fibonacci again! With $d(1) = 1, d(2) = 1$: $\ldots, 13, 21$, so $d(8) = 21 = F_8$.

> **Investigate further:** That compositions-into-odd-parts equals a Fibonacci number is startling until you find the bijection (group the parts cleverly, or use the recurrence's telescoping). Which Fibonacci? $d(n) = F_n$. Prove $d(n) = d(n-1) + d(n-2)$ directly by considering whether the last part is 1.

6. Length-6 strings over $\{0, 1, 2\}$ with no two adjacent 0s?

Answer: 448

Method: Two-state recurrence on the last symbol. Let $z(n)$ count valid strings ending in 0 and $w(n)$ those ending in 1 or 2. A 0 may follow only a non-0: $z(n) = w(n-1)$. A 1 or 2 may follow anything, in 2 ways: $w(n) = 2\big(z(n-1) + w(n-1)\big)$. Writing $a_n = z(n) + w(n)$ for the total gives $a_n = 2a_{n-1} + 2a_{n-2}$, seeded $a_1 = 3$, $a_2 = 8$: $3, 8, 22, 60, 164, 448$.

> **Investigate further:** The "$2a_{n-2}$" encodes the two non-zero symbols that can bridge a forbidden pair — coefficients count local choices, exactly as in C2. Predict the recurrence for a k-letter alphabet with no two adjacent 0s ($a_n = (k-1)a_{n-1} + (k-1)a_{n-2}$), and check it collapses to Fibonacci at $k = 2$.

7. Domino tilings of a 3×4 grid?

Answer: 11

Method: Odd-width strips need a genuine transfer-matrix (column-state) recurrence rather than a two-term one; carrying the profile of filled cells column by column yields 11. (The $3 \times n$ tiling counts run $1, 3, 11, 41, 153, \ldots$, satisfying $a_n = 4a_{n-1} - a_{n-2}$.)

> **Investigate further:** Note $3 \times n$ needs n even to be tileable at all — a parity/colouring invariant (colour the 12 cells like a chessboard: 6 of each, and each domino covers one of each). The recurrence $a_n = 4a_{n-1} - a_{n-2}$ hides in the transfer matrix; find its characteristic equation.

8. Binary strings of length 10 with no three consecutive 1s?

Answer: 504

Method: State = number of 1s currently at the end (0, 1, or 2). A valid string of length n ends in 0, in one trailing 1, or in two: summing gives $h(n) = h(n-1) + h(n-2) + h(n-3)$ (tribonacci), seeded $h(1) = 2, h(2) = 4, h(3) = 7$. Running to $n = 10$: $\ldots, 274, 504$.

> **Investigate further:** "No three in a row" widens the memory to the last two symbols — a three-term recurrence, exactly as "no two in a row" gave a two-term one. Predict the recurrence for "no *four* consecutive 1s" before deriving it (tetranacci). The forbidden-run length sets the recurrence order.

Section D Challenge (BMO1 difficulty)

1. Permutations with $|p(i) - i| \leq 1$: prove $a_n = a_{n-1} + a_{n-2}$; find a_6.

Answer: $a_6 = 13$

Method: *Proof.* Consider where n is sent. Since $|p(n) - n| \leq 1$, either $p(n) = n$ or $p(n) = n - 1$. *Case $p(n) = n$:* the restriction of p to $\{1, \ldots, n-1\}$ is any valid permutation of that set, giving a_{n-1} possibilities. *Case $p(n) = n - 1$:* then some i has $p(i) = n$, and $|i - n| \leq 1$ forces $i = n - 1$, so $p(n-1) = n$. Thus n and $n - 1$ are swapped, and p restricted to $\{1, \ldots, n-2\}$ is any valid permutation: a_{n-2} possibilities. The cases are disjoint and exhaustive, so $a_n = a_{n-1} + a_{n-2}$. With $a_1 = 1$, $a_2 = 2$ we get $1, 2, 3, 5, 8, 13$, so $a_6 = 13$. $\square$

> **Investigate further:** The proof's engine is "look at the most constrained element (n) and split on its few options" — the surest way to *find* a recurrence. These permutations biject with $2 \times n$ tilings (a fixed point = vertical domino, a swap = horizontal pair): so $a_n = F_{n+1}$, and D1 was B3 in disguise all along.

2. Erase two numbers, write their difference, repeat: prove the last number is odd.

Answer: Proof below.

Method: *Proof (parity invariant).* Track the parity of the sum S of all numbers on the board. Replacing a and b by $|a-b|$ changes the sum by $|a-b|-(a+b)=-2\min(a,b)$, an even number. So $S \bmod 2$ never changes — it is an *invariant.* Initially $S=1+2+\cdots+10=55$, which is odd. When one number remains it equals S, whose parity is still odd; hence the final number is odd. □

> **Investigate further:** The whole proof is "find a quantity every move preserves." Note $|a-b|$ and $a+b$ have the same parity — that is why the operation is parity-safe. Investigate the reachable *values*, not just parity: what is the largest possible final number? The smallest? (The gcd of all the numbers bounds what survives.)

3. Prove a convex hexagon has 14 triangulations.

Answer: $C_4 = 14$

Method: *Proof.* Let T_n be the number of triangulations of a convex $(n+2)$-gon; we show $T_n = C_n$, the Catalan number. Fix the edge AB of the polygon. In any triangulation, AB lies in exactly one triangle ABX for some other vertex X. That triangle splits the polygon into a smaller polygon on one side of AX (with $i+2$ vertices) and another on the side of XB (with $n-i+1$ vertices), each independently triangulated. Summing over the position of X gives $T_n=\sum_{i=0}^{n-1} T_i T_{n-1-i}$ — the Catalan recurrence, with $T_0=1$. Hence $T_n=C_n$; for a hexagon $n=4$ and $C_4=\frac{1}{5}\binom{8}{4}=14$. □

> **Investigate further:** The "fix an edge, split at its triangle" move is the canonical Catalan recurrence, shared with bracket-matching (A7) and Dyck paths (C3). Verify $C_4=14$ against the closed form and, if you have patience, by careful listing. Which other objects does $C_4=14$ count? (Binary trees with 4 internal nodes, among many.)

4. Prove any $2^n \times 2^n$ board minus one square is L-tromino tileable.

Answer: Proof below (induction).

Method: *Proof by induction on n. Base $n=1$:* a 2×2 board with one square removed is exactly a single L-tromino. *Step:* assume every $2^{n-1}\times 2^{n-1}$ board with one square removed can be tiled. Given a $2^n \times 2^n$ board with one square missing, cut it into four $2^{n-1}\times 2^{n-1}$ quadrants. The missing square lies in one quadrant. Place one L-tromino at the centre covering one corner square from each of the *other three* quadrants. Now every quadrant has exactly one square unavailable (the genuinely missing one, or the centre corner just covered), so by the inductive hypothesis each quadrant can be tiled. Together with the central tromino this tiles the whole board. □

> **Investigate further:** This is *constructive induction*: the proof is an algorithm you could run. It works precisely because $2^n \times 2^n$ splits into four self-similar quarters — the same divide-and-conquer as merge sort. Note $\frac{4^n-1}{3}$ trominoes are used; check this is an integer for all n (it must be — why?). Does the theorem hold for a 6×6 board minus one square? (No self-similarity — investigate.)

5. Remove a power of 2 $(1, 2, 4, 8, \ldots)$ from 2025; last stone wins. Who wins?

Answer: The second player.

Method: *Proof.* Call a position (the number of stones the mover faces) *losing* if the mover loses under optimal play; we claim the losing positions are exactly the multiples of 3. The key fact is that *no power of 2 is divisible by 3*: since $2\equiv -1 \pmod 3$, we have $2^k \equiv (-1)^k \equiv \pm 1 \pmod 3$, never 0. *From a multiple of 3:* subtracting any power of 2 (residue ±1) leaves a non-multiple of 3 — the mover cannot avoid handing the opponent a non-multiple. *From a non-multiple of 3:* the mover removes 1 stone (if the residue is 1) or 2 stones (if the residue is 2) — both are powers of 2 — reaching a multiple of 3. Since position 0 is losing, induction gives that the losing positions are precisely $0, 3, 6, \ldots$. As $2025 = 3 \times 675$ is a multiple of 3, the first player faces a losing position, so the **second** player wins by always restoring the pile to a multiple of 3. □

> **Investigate further:** The extra content over the plain 1-or-2 game is the modular fact $2^k \equiv (-1)^k \pmod 3$ — the large move set collapses to two useful residues. Now try removing any *Fibonacci* number of stones: the losing positions are no longer an arithmetic progression (Zeckendorf's theorem governs them) — a beautiful, much harder game to sit with.

Top 5 Patterns Today

1. **Hear the recurrence.** "The last move removed 1 or 2 units" means $f(n) = f(n-1) + f(n-2)$. Read the final choice, write the recurrence, seed it, run it.
2. **Carry the state.** A forbidden adjacency (no two 2-steps, no three 1s) forces you to remember the recent past: a multi-state or higher-order recurrence.
3. **Bijections prove equalities for free.** Stairs = tilings = no-11 strings, all Fibonacci; brackets = Dyck paths = triangulations, all Catalan. Exhibit the map, skip the formula.
4. **Invariants forbid.** A quantity preserved by every move (a parity, a colouring, a residue) proves some end-state is unreachable, no matter the strategy.
5. **Monovariants count and games decide.** A quantity that changes by a fixed step each move fixes the move count (snaps); a residue the winner can always restore names the winning strategy (Nim).

Common Traps to Avoid

1. **Trusting a doubling pattern.** $1, 2, 4, 8, 16$ then **31**, not 32 (C4). A sequence is not a formula until proved; five terms prove nothing.
2. **Wrong seed values.** A correct recurrence with the wrong $f(1), f(2)$ gives wrong answers forever. Compute the first two cases by hand, carefully.
3. **One-state recurrence for a two-state problem.** "No two 2-steps" cannot be counted by a single $f(n)$; you must track the last move. Under-modelling the state silently overcounts.
4. **Constructive claim without the construction.** D4 and B9 need an explicit tiling/strategy, not just "it works by induction/it's optimal." Build the object; markers award the algorithm.
5. **Naming the invariant but not checking every move preserves it.** State the quantity *and* verify each legal move leaves it unchanged — the parity of $|a-b|$ vs $a+b$ in D2 is the crux, not a footnote.

"A recurrence remembers where it came from; an invariant refuses to forget. Between them they count almost everything."

Found a mistake or want to contribute a sheet?

Visit the open-source project at github.com/The-CerealDev/speed-maths