Daily Combinatorics Drill #6 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Pigeonhole Principle (basic & strong) · Worst-Case Thinking · Double Counting · The Handshake Lemma.

Section A Rapid Recognition

($\leq$15 s each)

1. Socks to guarantee a matching pair?

Answer: 3

Method: Worst case: one of each colour (2 socks). One more forces a match.

> **Investigate further:** The answer is (number of colours) +1, independent of how many socks the drawer holds. What DOES the drawer size affect? (Whether the guarantee is achievable at all — with one black sock total, no black pair exists.)

2. Draws to guarantee a black pair?

Answer: 12

Method: Worst case: all 10 white socks first, then 2 black.

> **Investigate further:** "A pair" vs "a *black* pair" changed the answer from 3 to 12 — adversarial drawing exhausts everything unhelpful first. Rule: to guarantee X, assume the drawer fights you optimally, then add what X still needs.

3. 13 people, two share a birth month?

Answer: True

Method: 13 pigeons, 12 month-boxes.

> **Investigate further:** The principle is an *existence* claim: it never says which month. That non-constructive character is what makes pigeonhole proofs feel like magic — and what D-section write-ups must state cleanly.

4. Guarantee two share a weekday of birth?

Answer: 8

Method: 7 boxes, so 8 people.

> **Investigate further:** General basic form: k boxes need $k+1$ pigeons. Say it as a sentence: "if $k+1$ objects go into k boxes, some box holds two." Being able to *state* the principle precisely is worth marks in proofs.

5. Integers to guarantee two of the same parity?

Answer: 3

Method: Two parity boxes.

> **Investigate further:** Same-parity difference is even — so among any 3 integers, two differ by an even number (A10 is the 4-integer version — why does it also hold with just 3?).

6. Handshake counts total 18: how many handshakes?

Answer: 9

Method: Each handshake is counted twice, once by each participant: 18/2.

> **Investigate further:** The Handshake Lemma: $\sum \deg = 2 \cdot (\text{edges})$. Immediate corollary: the number of people with an ODD handshake count is even. Check it at your next gathering; prove it from the lemma in one line.

7. Row totals sum = column totals sum?

Answer: True

Method: Both equal the sum of every entry — one grid, two reading orders.

> **Investigate further:** This trivial-looking fact IS double counting. All of today's incidence arguments (B10, C6, D4) are this sentence applied to a cleverly chosen grid: rows = people, columns = committees, entries = memberships.

8. Guarantee two integers differing by a multiple of 5?

Answer: 6

Method: Boxes are the residues mod 5; same residue ⇒ difference divisible by 5.

> **Investigate further:** Residue classes are the most important pigeonholes in competition maths. Note the construction side: $1, 2, 3, 4, 5$ (one per residue) shows 5 integers do NOT suffice — a guarantee question is only fully answered when both directions are shown.

9. Sweets to guarantee 3 of one flavour (4 flavours)?

Answer: $4 \cdot 2 + 1 = 9$

Method: Worst case: 2 of every flavour (8 sweets); the ninth forces a third somewhere.

> **Investigate further:** Strong pigeonhole: to force m in one of k boxes you need $k(m-1)+1$ pigeons. Re-derive A1 ($m = 2$, $k = 2$) and C1 from this one formula.

10. Any 4 integers: two with even difference?

Answer: True

Method: Two parity boxes, four integers — some two share parity, and their difference is even.

> **Investigate further:** Sharper: among any 3 integers two share parity (A5). Among any 4 integers, must two differ by a multiple of 3? (Yes — 3 boxes.) The box count, not the pigeon count, is the quantity to engineer.

Section B Manipulation Drills ($\leq$50 s each)

1. 12 round-table seats: smallest number forcing an adjacent pair? *(after TMUA 2019 P2 Q9)*

Answer: 7

Method: Six diners can sit in alternate seats (construction: seats $1, 3, 5, 7, 9, 11$), so 6 does not force adjacency. With 7 diners, pair the seats into 6 adjacent couples $\{1, 2\}, \{3, 4\}, \ldots, \{11, 12\}$: seven diners in six couples put two in the same couple — adjacent. Answer: 7.

> **Investigate further:** Both halves matter: the construction shows 6 fails, the pigeonhole shows 7 works. TMUA multiple-choice hides this structure; BMO1 demands you write both. Generalise: $2n$ seats need $n + 1$ diners.

2. Guarantee two integers with difference divisible by 7?

Answer: 8

Method: Seven residue boxes mod 7.

> **Investigate further:** Follow-up with structure: among any 8 integers, two differ by a multiple of 7 — but among any *four* integers, must two SUM or differ to a multiple of 5? (Boxes $\{0\}, \{1, 4\}, \{2, 3\}$ mod 5: three boxes, four pigeons — yes. Pairing residues that sum to 5 is the C2 trick in modular clothing.)

3. Two of any 5 integers differ by a multiple of 4; five integers avoiding multiples-of-5 differences.

Answer: Pigeonhole on 4 residues; construction: $1, 2, 3, 4, 5$

Method: Mod 4 there are only 4 residues, so 5 integers repeat one; equal residues difference $\equiv 0$. For the second part, any complete residue system mod 5 works: $1, 2, 3, 4, 5$ have all differences 1–4 mod 5, never 0.

> **Investigate further:** Notice the numbers 4 and 5 trade places in the two halves — the guarantee threshold for modulus m is exactly $m + 1$. State this as a single theorem covering A8, B2 and this question.

4. Six from $\{1, \ldots, 10\}$: two summing to 11? *(after TMUA 2022 P2 Q8)*

Answer: Yes; the smallest forcing size is 6

Method: Pair the set into $\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}$: five boxes each summing to 11. Six chosen numbers land two in one box. Five numbers can dodge (take one from each pair, e.g. $1, 2, 3, 4, 5$).

> **Investigate further:** Sum-pairs are pigeonholes just as residues are. Design your own: from $\{1, \ldots, 20\}$, how many numbers force two summing to 21? Two differing by 10? (Same boxes both times! Why?)

5. Round-robin with 8 teams: matches and per-team games?

Answer: $\binom{8}{2} = 28$ matches; each team plays 7

Method: A match is a pair of teams. Per team: everyone else once.

> **Investigate further:** Double-count check: $8 \cdot 7$ counts each match twice, $56/2 = 28$. ✓ Real leagues

> double it (home and away: 56). How many matches in a league where each team plays each other k times?

6. Friendship counts total 84: how many friendships?

Answer: 42

Method: Handshake lemma: each friendship contributes 2 to the total.

> **Investigate further:** Could the counts have totalled 85? (No — the degree sum is always even.) This parity fact silently solves impossible-configuration questions: e.g. can 7 people each have exactly 3 friends? ($7 \cdot 3 = 21$ odd — impossible.)

7. Four from $\{1, \dots, 6\}$: two consecutive?

Answer: Forced; largest safe choice has 3 elements, e.g. $\{1, 3, 5\}$

Method: Boxes $\{1, 2\}, \{3, 4\}, \{5, 6\}$: four numbers in three boxes put two in one box, and box-mates are consecutive. The construction $\{1, 3, 5\}$ (or $\{2, 4, 6\}$, $\{1, 4, 6\}$, $\dots$) shows 3 can dodge.

> **Investigate further:** Connect back to sheet 2's D2: subsets of $\{1, \dots, 8\}$ with NO two consecutive were counted by Fibonacci (55 of them). Guarantee questions ask for the *largest* such subset instead — extremal vs enumerative, two faces of one constraint.

8. Two party guests with equal handshake counts: why can't 0 and $n - 1$ coexist?

Answer: Someone shaking all $n-1$ hands has shaken the non-shaker's hand — contradiction.

Method: Possible counts are $0, 1, \dots, n - 1$: that is n boxes for n people, so pigeonhole alone fails. But the extreme boxes are incompatible: a count of $n - 1$ (shook everyone) and a count of 0 (shook no one) cannot both occur. So at most $n - 1$ counts are actually available for n people — pigeonhole now bites.

> **Investigate further:** This "shrink the boxes by a contradiction" move upgrades many pigeonhole arguments. D3 asks for the full formal write-up — draft it now while the idea is warm.

9. 25 sweets, 7 children: someone gets at least 4.

Answer: At least $\lceil 25/7 \rceil = 4$

Method: If every child had at most 3, the total would be at most $21 < 25$. Contradiction.

> **Investigate further:** The averaging form of pigeonhole: some value is $\geq$ the mean (and some is $\leq$). Convert to the strong form: $7 \cdot 3 + 1 = 22$ sweets already force a 4; we had 25. Which formulation writes up faster?

10. 5 committees of 4; everyone on exactly 2: how many members?

Answer: $\dfrac{5 \cdot 4}{2} = 10$

Method: Count (person, committee) memberships two ways: committees give $5 \cdot 4 = 20$; people give $2 \cdot$ (members). So members = 10.

> **Investigate further:** A7's grid made explicit: rows are people, columns committees. Whenever a puzzle says "each X has a Y's and each Y has b X's", the incidence count $a|X| = b|Y|$ falls out. D4 is this with one unknown.

Section C Substitution & Structure (≤90 s each)

1. 10 red, 10 green, 10 blue: guarantee 5 of some colour; guarantee 5 red?

Answer: Some colour: $3 \cdot 4 + 1 = 13$. Five red: $20 + 4 + 1 = 25$.

Method: *Some colour* is plain strong pigeonhole: worst case 4 of each (12 beads), the 13th forces a fifth. *Five red* is adversarial: the drawer hands you all 10 green and 10 blue first (20 useless beads), then 4 red — so 24 can still fail, and the 25th guarantees a fifth red.

> **Investigate further:** The two answers differ because "some colour" lets any colour win, while "red" must survive the drawer burning every other colour first — the A2 lesson (a pair vs a black pair) scaled up. What if you needed 5 red *and* 5 green? Chase the worst case: burn blue first.

2. Seven from $\{1, \ldots, 12\}$: two sum to 13; six don't force.

Answer: Forced at 7; construction for 6: $\{1, 2, 3, 4, 5, 6\}$

Method: Boxes $\{1, 12\}, \{2, 11\}, \{3, 10\}, \{4, 9\}, \{5, 8\}, \{6, 7\}$ — six boxes each summing to 13. Seven numbers double up somewhere. Taking the smaller element of each pair (1–6) dodges with six numbers: all pair-sums are at most $11 < 13$.

> **Investigate further:** Verify the construction really works (largest sum $5 + 6 = 11$). Then a two-step hybrid: how many numbers from $\{1, \ldots, 12\}$ force EITHER a pair summing to 13 OR a pair differing by 6? (The boxes coincide! Explain why $\{k, 13 - k\}$ and difference-6 pairs interlock.)

3. Guarantee two integers with the same last digit?

Answer: 11

Method: Ten digit-boxes 0–9.

> **Investigate further:** Same last digit $\iff$ difference divisible by 10 — so this is A8/B2 with modulus 10. Chain it: among 11 integers, two differ by a multiple of 10; among how many, two differing by a multiple of 100? The modulus is the box count, always.

4. Mutilated chessboard: 31 dominoes?

Answer: Impossible.

Method: Colour the board alternately. Opposite corners share a colour, so the mutilated board has 32 squares of one colour and 30 of the other. Every domino covers one square of each colour, so 31 dominoes cover 31 of each — but $31 \neq 32$. No tiling exists. □

> **Investigate further:** The first *invariant* argument of the pillar: a quantity every move preserves (colour balance) separates reachable from unreachable. Sheet 7 builds a whole section on this. Try: remove two squares of OPPOSITE colours anywhere — can the remaining 62 always be tiled? (Yes — Gomory's beautiful cycle argument; look it up after attempting.)

5. Group size to guarantee two share both birth month and weekday?

Answer: $12 \cdot 7 + 1 = 85$

Method: The pigeonholes are (month, weekday) pairs: $12 \times 7 = 84$ boxes. With 85 people, two land in the same box — same month AND same weekday.

> **Investigate further:** Two independent attributes multiply their box counts, they do not add — 84, not 19. This product-of-dimensions idea reappears whenever you pigeonhole on several coordinates at once (residues mod different primes, grid cells). How many boxes if you also track blood type (4 kinds)?

6. 20 clubs $\times$ 6 students; each student in 3 clubs: student count?

Answer: $\dfrac{20 \cdot 6}{3} = 40$

Method: Memberships counted by clubs: 120. Counted by students: $3S$. So $S = 40$.

> **Investigate further:** Add a consistency question: could the same school run 21 clubs of 6 with everyone in exactly 3? ($126/3 = 42$ — fine). And 20 clubs of 7? ($140/3$ is not an integer — impossible!). Divisibility of the double count is a hidden feasibility test.

7. 1–9 around a circle: three adjacent summing ≥ 15.

Answer: Some three adjacent numbers sum to at least 15.

Method: The nine positions split into three *disjoint* blocks of three consecutive seats. The three block-sums total $1 + 2 + \cdots + 9 = 45$, so some block sums to at least $\lceil 45/3 \rceil = 15$. □

> **Investigate further:** Averaging pigeonhole on engineered boxes. Sharper (harder): show some three adjacent sum ≥ 16 fails — find an arrangement whose every adjacent triple sums ≤ 16 but... actually can every triple be ≤ 15? The nine *overlapping* triples sum to $3 \cdot 45 = 135 = 9 \cdot 15$ exactly — so all nine triples equal 15, forcing impossible structure. Chase this to decide whether 15 is tight.

8. Five from $\{1, \ldots, 8\}$: two differing by 4.

Answer: Forced.

Method: Boxes $\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}$: four boxes whose mates differ by exactly 4. Five numbers put two in one box. □

> **Investigate further:** Largest safe set: 4 (one per box — e.g. $\{1, 2, 3, 4\}$). Variant with difference 3 on $\{1, \ldots, 9\}$: boxes $\{1, 4\}, \{2, 5\}, \{3, 6\}, \{7, ?\}$... the pairing frays. Build correct boxes for difference 3 (chains $1 \to 4 \to 7$, $2 \to 5 \to 8$, $3 \to 6 \to 9$: pick alternate links) and find the true threshold.

Section D Challenge

(SMC / BMO1 difficulty)

1. Five points in a unit square: two within $\frac{\sqrt{2}}{2}$.

Answer: Proof below.

Method: *Proof.* Divide the unit square into four closed $\frac{1}{2} \times \frac{1}{2}$ quarter-squares. Five points in four quarters: by the pigeonhole principle two points, say P and Q, lie in the same quarter. Any two points of a $\frac{1}{2} \times \frac{1}{2}$ square are at distance at most its diagonal, $\frac{\sqrt{2}}{2}$. Hence $|PQ| \leq \frac{\sqrt{2}}{2}$. □

> **Investigate further:** The write-up has three named steps: build boxes, apply pigeonhole, bound within a box — every geometric pigeonhole proof has this skeleton. Is $\frac{\sqrt{2}}{2}$ best possible for five points? (Place four corners + centre and measure.) What about *four* points — what distance can they all keep?

2. Five lattice points: some midpoint is a lattice point.

Answer: Proof below.

Method: *Proof.* Classify each point (x, y) by the parities of its coordinates: (even, even), (even, odd), (odd, even), (odd, odd) — four classes. Five points give two in the same class, say $P = (a, b)$ and $Q = (c, d)$ with $a \equiv c$ and $b \equiv d \pmod 2$. Then $a + c$ and $b + d$ are both even, so the midpoint $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$ has integer coordinates. □

Investigate further: Four points can dodge: $(0, 0), (0, 1), (1, 0), (1, 1)$ — one per parity class; check all six midpoints fail. So 5 is exact. Generalise to 3D: how many lattice points force a lattice midpoint? $(2^3 + 1 = 9.)$ And what forces a lattice *centroid* of some three points? (Much subtler — sit with it.)

3. The number of people with an odd handshake count is even.

Answer: Proof below.

Method: *Proof.* Let the guests be the vertices of a graph, with an edge for each handshake; write $d(v)$ for the number of hands v shook. Every handshake has two participants, so summing over all guests counts each handshake twice: $\sum_v d(v) = 2H$, where H is the total number of handshakes — an even number. Split the sum by parity: $\sum_{d(v)\text{ even}} d(v)$ is even (a sum of even numbers), and the whole sum is even, so $\sum_{d(v)\text{ odd}} d(v)$ is even too. But that last sum is a sum of odd numbers, and a sum of odd numbers is even only when there is an even count of them. Hence the number of guests with an odd handshake count is even. □

Investigate further: This is the Handshake Lemma's famous corollary. It instantly kills impossible configurations: can 9 people each shake exactly 3 hands? (Nine odds is odd — no.) Sit with the parity step "a sum of odd numbers is even iff there is an even number of them" until it is obvious; it is the whole proof.

4. 20 students, each solving exactly 2 of 5 problems: two share a problem-pair.

Answer: Proof below; some pair chosen by ≥ 2 students (in fact $\geq \lceil 20/10 \rceil = 2$).

Method: *Proof.* Each student's choice of two problems from five is an unordered pair, and there are exactly $\binom{5}{2} = 10$ such pairs. Regard the 10 pairs as pigeonholes and the 20 students as pigeons. Since $20 > 10$, by the pigeonhole principle some pigeonhole holds at least two students — that is, some pair of problems was chosen by at least two students, who therefore solved exactly the same pair. (Strong pigeonhole sharpens this: at least $\lceil 20/10 \rceil = 2$ share the most popular pair, and averaging shows some pair is chosen by at least 2 regardless of the distribution.) □

Investigate further: Naming the pigeonholes as the $\binom{5}{2}$ pairs — rather than the problems — is the whole idea; a student who lists the wrong boxes gets nothing. Sharpen it: with 21 students must some pair be chosen three times? (No — $21 < 2 \cdot 10 + 1$.) Find the exact threshold that forces a triple.

5. Six people: three mutual friends or three mutual strangers.

Answer: Proof below.

Method: *Proof.* Fix a person X. The other five people split into friends of X and strangers to X; by pigeonhole one class contains at least $\lceil 5/2 \rceil = 3$ people — say (without loss of generality) A, B, C are all friends of X. Now examine the three relations among A, B, C: if any pair, say A and B, are friends, then X, A, B are three mutual friends. Otherwise all three pairs among A, B, C are strangers, and A, B, C are three mutual strangers. Either way the claim holds. (If the majority class was strangers, the same argument runs with roles swapped.) □

> **Investigate further:** This is the Ramsey number statement $R(3,3) \leq 6$. Show 5 people are not enough: arrange five people in a cycle where neighbours are friends and non-neighbours strangers, and verify no monochromatic triangle. Hence $R(3,3) = 6$ exactly. The write-up's "without loss of generality" is load-bearing — say why the symmetry justifies it.

Top 5 Patterns Today

1. **Basic pigeonhole.** $k+1$ objects in k boxes: some box holds two. The craft is in *designing* the boxes: residues, sum-pairs, quarter-squares, parity classes.
2. **Strong pigeonhole.** Forcing m alike among k types needs $k(m-1)+1$ objects. Equivalently: some box holds at least the ceiling of the average.
3. **Worst case first.** Every guarantee number is (largest configuration that dodges) +1. A complete answer exhibits the dodge AND proves the force.
4. **Double counting.** Count (X, Y) incidences by X and by Y: $a|X| = b|Y|$. Grids, committees, degree sums — and an integrality feasibility test for free.
5. **Shrink the boxes by contradiction.** When boxes equal pigeons, show two boxes cannot both be occupied (counts 0 and $n-1$) — then pigeonhole closes the argument.

Common Traps to Avoid

1. **Answering the force without the dodge.** "7 diners force adjacency" is half marks until you seat 6 who don't. Guarantee = construction + pigeonhole.
2. **Off-by-one in strong pigeonhole.** It is $k(m-1)+1$, not km or $km+1$. Re-derive from the worst case each time rather than recalling.
3. **Boxes that aren't disjoint or don't cover.** Sum-pair and difference boxes must partition the set — check both properties explicitly before counting pigeons.
4. **Double counting with one side wrong.** "Each of 20 clubs of 7, everyone in 3" has no integer solution — test divisibility before trusting a puzzle's premises.
5. **Vague write-ups.** "By pigeonhole, done" scores nothing. Name the pigeons, name the boxes, state which two collide and what that collision gives you.

"The pigeonhole principle proves existence without ever finding the thing — mathematics at its most confident."

Found a mistake or want to contribute a sheet?
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