Daily Combinatorics Drill #5 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Stars & Bars · Substitution Shifts · Inclusion–Exclusion · Counting with Caps.

Section A Rapid Recognition (≤15 s each)

1. Non-negative (x, y) with $x + y = 10$?

Answer: 11

Method: x can be $0, 1, \ldots, 10$; then y is forced.

> **Investigate further:** Trivial by listing — but note it is also $\binom{11}{1}$: ten stars, one bar. Every question on this sheet is this question with more bars.

2. 5 identical sweets to 3 children (zeros allowed)?

Answer: $\binom{7}{2} = 21$

Method: Stars and bars: 5 stars, 2 bars in a row of 7 symbols; choose the bar positions.

> **Investigate further:** Draw the correspondence explicitly for one case: $\star\star \mid \star \mid \star\star$ means $(2, 1, 2)$. The *bijection* between distributions and star–bar strings is the proof — rehearse saying it in one sentence.

3. $|A \cup B|$ from $15, 12, 5$?

Answer: $15 + 12 - 5 = 22$

Method: Two-set inclusion–exclusion: adding double-counts the overlap once, so subtract it once.

> **Investigate further:** Rearrange to find any missing quantity: $|A \cap B| = |A| + |B| - |A \cup B|$. TMUA logic questions like giving you three of the four numbers — be ready to solve for each.

4. Multiples of 3 up to 100?

Answer: $\lfloor 100/3 \rfloor = 33$

Method: Floor of the quotient.

> **Investigate further:** Why floor and not rounding? Count multiples of 3 in $1, \ldots, 101$ and in $1, \ldots, 102$ and watch the count tick up only at true multiples. This tiny function powers C4 and D3.

5. $|A| = 8$, $|B| = 6$, $|A \cup B| = 11$: find $|A \cap B|$.

Answer: $8 + 6 - 11 = 3$

Method: Rearrange two-set inclusion–exclusion: $|A \cap B| = |A| + |B| - |A \cup B|$.

> **Investigate further:** The formula has four quantities; any three determine the fourth. What constraints must the three given numbers satisfy for a solution to exist at all? ($|A \cup B| \leq |A| + |B|$ and $|A \cup B| \geq \max$ — data violating these describe no actual sets, a favourite TMUA logic trap.)

6. Non-negative triples, $x + y + z = 4$?

Answer: $\dbinom{6}{2} = 15$

Method: 4 stars, 2 bars.

> **Investigate further:** General formula: non-negative solutions of $x_1 + \cdots + x_k = n$ number $\binom{n+k-1}{k-1}$. Re-derive it from the star–bar picture, then never look it up again.

7. Positive triples, $x + y + z = 6$?

Answer: $\dbinom{5}{2} = 10$

Method: Shift: $x' = x - 1$ etc. gives non-negative $x' + y' + z' = 3$: $\binom{5}{2}$. Equivalently, place 2 bars in the 5 gaps between six stars.

> **Investigate further:** Two mental models — shift-then-stars vs bars-in-gaps. They must agree: $\binom{n-1}{k-1}$ both ways. Which generalises to "each variable at least 2"? (Both: shift by 2, or leave a star glued to each bar.)

8. Class of 30: 20 French, 15 German, 8 both — neither?

Answer: $30 - (20 + 15 - 8) = 3$

Method: Union first (27), then complement.

> **Investigate further:** Draw the Venn diagram and fill the four regions (12, 8, 7, 3). Region-filling is slower but incorruptible — use it to check any two-set I–E answer under exam pressure.

9. True or false: 7 has six ordered two-part sums?

Answer: True (1+6, 2+5, ..., 6+1)

Method: The first part can be 1 through 6; the second is forced.

> **Investigate further:** "Ordered sums" are *compositions*. Guess the count of ordered three-part sums of 7 ($\binom{6}{2} = 15$), then the total over all lengths — D2 proves the astonishing answer.

10. 4 identical marbles, 2 boxes?

Answer: 5

Method: The first box holds 0–4.

> **Investigate further:** As stars and bars: $\binom{5}{1} = 5$. ✓ Now make the marbles *distinct*: $2^4 = 16$. Identical-vs-distinct changes the formula family entirely — always classify the objects before counting.

Section B Manipulation Drills

($\leq$50 s each)

1. Non-negative triples, $x + y + z = 12$?

Answer: $\binom{14}{2} = 91$

Method: 12 stars, 2 bars: $\binom{14}{2}$.

> **Investigate further:** Sanity-check stars-and-bars answers by shrinking: the same formula must give 15 for $n = 4$ (A6). A formula you can shrink-test is a formula you can trust.

2. Positive triples, $x + y + z = 12$?

Answer: $\binom{11}{2} = 55$

Method: Shift each variable down by 1: non-negative sum 9, so $\binom{11}{2}$.

> **Investigate further:** Positive solutions of sum n = non-negative solutions of sum $n - k$. State the general bridge and apply it mentally: positive quadruples summing to 10? ($\binom{9}{3} = 84$.)

3. 10 sweets, 4 children, each at least one?

Answer: $\binom{9}{3} = 84$

Method: Bars-in-gaps: ten stars have 9 internal gaps; choose 3.

> **Investigate further:** Now demand each child gets at least *two*: shift by 2 each, sum drops to 2, count $\binom{5}{3} = 10$. Lower bounds are always shifts — upper bounds are the day's harder story (C2, C5).

4. 1–1000, divisible by neither 2 nor 5?

Answer: $1000 - 500 - 200 + 100 = 400$

Method: Complement of the union: subtract each, add back the multiples of 10.

> **Investigate further:** $400/1000 = (1 - \frac{1}{2})(1 - \frac{1}{5})$ — the count factorises! This is no accident: divisibility by 2 and by 5 are independent events across a full period. D3 turns this into Euler's φ.

5. Non-negative, $x + y + z = 10$, $x \geq 2$?

Answer: $\binom{10}{2} = 45$

Method: Shift only x: $x' = x - 2$ gives $x' + y + z = 8$: $\binom{10}{2}$.

> **Investigate further:** Mixed bounds bow to per-variable shifts: solve $x \geq 2$, $y \geq 1$, $z \geq 0$, sum 10 ($\binom{9}{2} = 36$). Write the shifted equation every time — skipping the substitution line is where errors breed.

6. Arrangements of $1, 2, 3, 4$ with 1 not first?

Answer: $4! - 3! = 18$

Method: Complement: total minus those with 1 first.

> **Investigate further:** Direct count: 3 choices of first digit, then 3! arrangements — 18 again. This tiny warm-up is one forbidden position; C6 has two, D1 has *all* positions forbidden. Watch the technique escalate.

7. 1–100, divisible by 2 or 3?

Answer: $50 + 33 - 16 = 67$

Method: Overlap is multiples of 6: $\lfloor 100/6 \rfloor = 16$.

> **Investigate further:** Check by complement: coprime-to-6 residues are 1 and 5 mod 6, giving $17 + 16 = 33$ non-multiples; $100 - 33 = 67$. ✓ Complement checks are nearly free — build the reflex.

8. Ordered three-part sums of 8?

Answer: $\binom{7}{2} = 21$

Method: Compositions of 8 into 3 positive parts: bars in 2 of the 7 gaps.

> **Investigate further:** Unordered partitions of 8 into 3 parts number only 5 (6+1+1, 5+2+1, 4+3+1, 4+2+2, 3+3+2). Why does dividing 21 by 3! NOT give 5? (Repeated parts break the symmetry — the same trap as sheet 2's "divide by symmetry that isn't there".)

9. Non-negative (x, y) with $x + y \leq 5$?

Answer: $\binom{7}{2} = 21$

Method: Slack variable: introduce $s = 5 - x - y \geq 0$; then $x + y + s = 5$, a three-variable stars-and-bars.

> **Investigate further:** The slack-variable move turns every inequality into an equation. Geometrically you counted lattice points in a triangle — check: $1 + 2 + \cdots + 6 = 21$. Two viewpoints, one number.

10. 5-letter A/B strings using both letters?

Answer: $2^5 - 2 = 30$

Method: Complement: only the all-A and all-B strings fail.

> **Investigate further:** This is inclusion–exclusion on 2 "missing letter" events in miniature. With three letters (strings over {A,B,C} using all three) the same logic needs the full machine: $3^5 - 3 \cdot 2^5 + 3 = 150$ — which is D4 in disguise. Peek ahead.

Section C Substitution & Structure (≤90 s each)

1. Positive quadruples, $x + y + z + w = 15$, none exceeding 5?

Answer: 52

Method: Both bound types at once. Unrestricted positive solutions: $\binom{14}{3} = 364$ (bars in gaps). Violations are variables ≥ 6: shift one variable by 5 (positive ≥ 6 becomes positive ≥ 1), leaving sum

10: $\binom{9}{3} = 84$, times 4 choices: 336. Double violations (two variables ≥ 6, sum drops to 5): $\binom{4}{3} = 4$ each, $\binom{4}{2} = 6$ pairs: 24. Triples impossible. Count: $364 - 336 + 24 = 52$.

> **Investigate further:** Check plausibility: the mean share is $15/4 = 3.75$ with cap 5 — tight, so few solutions survive; 52 of 364 feels right. Sanity-check one corner: $(5, 5, 4, 1)$ counts, $(6, 4, 4, 1)$ does not. When the cap equals the mean exactly, how many survive? (Sum 20, cap 5: only $(5, 5, 5, 5)$ — run the I–E and watch it collapse to 1.)

2. Non-negative triples, $x + y + z = 15$, each ≤ 6?

Answer: 10

Method: Inclusion–exclusion on the *violations*. Unrestricted: $\binom{17}{2} = 136$. Violating $x \geq 7$: shift by 7, sum 8: $\binom{10}{2} = 45$, and 3 choices of violator: 135. Two violators (≥ 7 twice, sum drops to 1): $\binom{3}{2} = 3$ each, 3 pairs: 9. Three violators impossible. Count: $136 - 135 + 9 = 10$.

> **Investigate further:** Caps are complements of shifts — the day's central lesson. Check the answer directly: with all variables ≤ 6 and sum 15, the average is 5, so solutions cluster near $(6, 6, 3)$-type triples; list them (permutations of $(6, 6, 3)$, $(6, 5, 4)$, $(5, 5, 5)$: $3 + 6 + 1 = 10$). ✓

3. Tea/coffee/juice: how many like none?

Answer: $100 - 80 = 20$

Method: Three-set I–E: $|T \cup C \cup J| = 50 + 40 + 30 - 20 - 15 - 10 + 5 = 80$.

> **Investigate further:** Recite the sign rule: singles − pairs + triple. Then answer a sharper question from the same data: how many like *exactly one* drink? (Singles minus twice-pairs plus thrice-triple: $120 - 2 \cdot 45 + 3 \cdot 5 = 45$.) Exactly-counts are I–E's second gear.

4. 1–1000, divisible by 2, 3, or 5?

Answer: 734

Method: $500 + 333 + 200 - 166 - 100 - 66 + 33 = 734$. The overlaps are lcm's: 6, 10, 15, 30.

> **Investigate further:** Complement: 266 integers survive — and $1000 \cdot (1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5}) = 266.67$, close but NOT exact. Why does the product formula miss here when it was exact in B4? (1000 is not a multiple of 30 — partial periods leave remainders. D3 chooses its numbers to dodge this.)

5. 8 apples, 3 children, each at most 4?

Answer: 15

Method: Unrestricted: $\binom{10}{2} = 45$. One child ≥ 5: shift 5, sum 3: $\binom{5}{2} = 10$, times 3 children: 30. Two children ≥ 5 would need sum $\geq 10 > 8$: impossible. Answer $45 - 30 = 15$.

> **Investigate further:** The "second violation impossible" step is what keeps caps quick — always bound the double-violation before computing it. Redo with cap 3 (each child ≤ 3): now double violations survive; the count is $45 - 3 \cdot \binom{6}{2} + 3 \cdot \binom{2}{2} = 3$. List them to believe it ($(3, 3, 2)$ permutations).

6. ABCDE arrangements: A not first, B not second?

Answer: $120 - 24 - 24 + 6 = 78$

Method: I–E on the two bad events (A first: 4!; B second: 4!; both: 3!).

> **Investigate further:** Escalate to three pinned letters (A not 1st, B not 2nd, C not 3rd): $120 - 3 \cdot 24 + 3 \cdot 6 - 2! = 64$. The pattern of shrinking factorials with alternating signs is the derangement formula being born — D1 delivers it.

7. Three dice summing to 10?

> **Answer:** 27

Method: Stars and bars with caps. Shift to $x' + y' + z' = 7$ with each $x' \in [0, 5]$: unrestricted $\binom{9}{2} = 36$; one variable ≥ 6: sum 1, $\binom{3}{2} = 3$, times 3: 9. Answer $36 - 9 = 27$.

> **Investigate further:** Dice sums are capped stars-and-bars — this exact question (sum of three dice) launched probability theory in the 1650s: compare the counts for sum 10 (27) and sum 9 (25), whose near-equality puzzled gamblers. Recompute the 25.

8. Positive integers below 1000 with digit sum 12?

> **Answer:** $\binom{14}{2} - 3\binom{4}{2} = 91 - 18 = 73$

Method: Write each number as a 3-digit string $d_1d_2d_3$ with leading zeros allowed: need $d_1 + d_2 + d_3 = 12$, digits 0–9. Unrestricted: $\binom{14}{2} = 91$. Violations $d_i \geq 10$: shift by 10, sum 2 remains: $\binom{4}{2} = 6$, times 3 digits: 18. Two violations impossible ($\geq 20 > 12$). Count: $91 - 18 = 73$.

> **Investigate further:** For digit sum 9 the cap never bites (no digit can reach 10) and the count is a pure $\binom{11}{2} = 55$. Find the digit sum with the MOST three-digit representatives — symmetry of the distribution around 13.5 says compare sums 13 and 14; compute both. The "pad with leading zeros" recoding is a bijection worth naming in written solutions.

Section D Challenge (SMC / BMO1 difficulty)

1. Show exactly 9 of the 24 letter placements have no letter correct.

> **Answer:** $D_4 = 9$

Method: *Proof.* For $i = 1, \ldots, 4$ let A_i be the set of placements with letter i correct. Placements fixing a given set of k letters number $(4-k)!$, and there are $\binom{4}{k}$ such sets, so by inclusion–exclusion $|A_1 \cup A_2 \cup A_3 \cup A_4| = \binom{4}{1}3! - \binom{4}{2}2! + \binom{4}{3}1! - \binom{4}{4}0! = 24 - 12 + 4 - 1 = 15$. Hence the placements with *no* letter correct number $24 - 15 = 9$. $\square$

> **Investigate further:** These are *derangements*: $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$. Compute $D_5 = 44$ and check $\frac{D_n}{n!} \to \frac{1}{e} \approx 0.368$ — over a third of all shuffles fix nothing, however large n grows. Investigate the elegant recurrence $D_n = (n-1)(D_{n-1} + D_{n-2})$.

2. Prove: compositions of n number 2^{n-1}.

> **Answer:** Proof below.

Method: *Proof (bijection with gap subsets).* Write n as a row of n stars. A composition of n corresponds to choosing, independently for each of the $n - 1$ gaps between adjacent stars, whether to place a bar there: the bars cut the row into ordered positive parts, and every composition arises from exactly one set of cuts. Choices: 2^{n-1}. Since compositions $\leftrightarrow$ subsets of the gap set, the count is 2^{n-1}. $\square$

> **Investigate further:** Cross-check against the by-length sum: $\sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}$ — the row-sum

> identity of sheet 4 falling out of a partition of the composition set by length. One object, two counts, one identity: this is what "combinatorial proof" means on a BMO script.

3. Integers 1–360 coprime to 360?

Answer: 96

Method: $360 = 2^3 \cdot 3^2 \cdot 5$, so remove multiples of 2, 3, 5 by inclusion–exclusion: $360 - 180 - 120 - 72 + 60 + 36 + 24 - 12 = 96$. Equivalently $360\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = 360 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 96$ — the product form is the I–E sum factorised.

> **Investigate further:** This is Euler's totient $\varphi(360)$. Prove the factorised form follows from I–E for any n (each prime contributes an independent "survival factor" $1 - \frac{1}{p}$ over a full period — B4's observation made general). φ returns as a lead actor in the number-theory pillar.

4. Five students, three clubs, none empty: show 150.

Answer: $3^5 - 3 \cdot 2^5 + 3 \cdot 1^5 = 150$

Method: *Proof.* Unrestricted assignments: $3^5 = 243$. For each club c let A_c be the assignments leaving c empty; $|A_c| = 2^5$, $|A_c \cap A_{c'}| = 1^5$, and all three clubs empty is impossible. By inclusion–exclusion the assignments missing at least one club number $3 \cdot 2^5 - 3 \cdot 1 = 93$, so surjective assignments number $243 - 93 = 150$. $\square$

> **Investigate further:** General surjection count: $\sum_k (-1)^k \binom{m}{k}(m-k)^n$ for n students, m clubs. Compute the 4-club version for 6 students (1560). Dividing by $m!$ counts *partitions into unlabelled groups* — the Stirling numbers, waiting one sheet ahead.

5. Prove exactly 998 910 of 1–10^6 are neither squares nor cubes.

Answer: $10^6 - 1000 - 100 + 10 = 998\,910$

Method: Squares up to 10^6: exactly 1000 (namely k^2 for $k \le 10^3$). Cubes: 100 (since $100^3 = 10^6$). The double-counted numbers are those that are both square and cube, i.e. sixth powers: 10 of them (k^6, $k \le 10$). By inclusion–exclusion, squares-or-cubes number $1000 + 100 - 10 = 1090$; the rest number $10^6 - 1090 = 998\,910$. $\square$

> **Investigate further:** Why is "both square and cube" the same as "sixth power"? (If $n = a^2 = b^3$, look at each prime's exponent: it is even and divisible by 3, hence divisible by 6.) That little lemma is the question's real content — the counting is decoration. Extend: how many are none of square, cube, or fifth power?

Top 5 Patterns Today

1. **Stars and bars.** Non-negative solutions of $x_1 + \cdots + x_k = n$: $\binom{n+k-1}{k-1}$. Positive solutions: bars in gaps, $\binom{n-1}{k-1}$.
2. **Shift for lower bounds.** "$x \ge c$" $\Rightarrow$ substitute $x' = x - c$ and reduce the target sum. Write the substitution line explicitly.
3. **Include–exclude for upper bounds.** Caps are violations to subtract: count unrestricted, remove each violation (shift by cap+1), add back double violations. Check when doubles are impossible.
4. **Slack variables.** "$\le n$" becomes "$= n$" with one extra non-negative variable. Inequalities are

equations in disguise.

5. **The I–E sign rhythm.** Singles − pairs + triples $- \cdots$, with overlaps measured by lcm's (divisibility) or forced objects (fixed letters, empty clubs).

Common Traps to Avoid

1. **Muddling the two stars-and-bars formulas.** Zeros allowed: $\binom{n+k-1}{k-1}$. Positive: $\binom{n-1}{k-1}$. Shrink-test on a tiny case before trusting either.

2. **Using the product instead of the lcm for overlaps.** Divisible by 4 and 6 means divisible by 12, not 24. Compute the lcm consciously every time.

3. **Forgetting the cap can't always bite.** In C8 no digit could exceed 9; in C5 no two children could both violate. Bound the violation before machining through it.

4. **Dividing compositions by $k!$ to get partitions.** Repeated parts have smaller orbits — 21 compositions of 8 into 3 parts give 5 partitions, not 21/6. Symmetry division needs distinct parts.

5. **Stopping I–E one term early.** The alternating sum runs until intersections are empty — and you must *say* why they are empty (sum too large, all clubs empty impossible), not just stop.

"Overcount boldly, then subtract with precision — inclusion–exclusion is bookkeeping for the brave."

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths