Daily Combinatorics Drill #4 — Answers & Investigations

Sheet by David Akinyele-Aje

Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Binomial Theorem · Pascal's Rule · Coefficient Extraction · Substituting Values into Expansions.

Section A Rapid Recognition (≤15 s each)

1. Coefficient of x^2 in $(1+x)^5$.

Answer: $\binom{5}{2} = 10$

Method: Binomial theorem: the x^k coefficient in $(1+x)^n$ is $\binom{n}{k}$.

> **Investigate further:** *Why* is it $\binom{5}{2}$? Expanding $(1+x)(1+x)(1+x)(1+x)(1+x)$, an x^2 term arises by choosing the x from exactly 2 of the 5 brackets. The binomial theorem *is* a counting statement — that lens powers everything on this sheet.

2. Coefficient of x in $(x+2)^3$.

Answer: 12

Method: Term with x^1: $\binom{3}{1}x \cdot 2^2 = 12x$. Keep track of which quantity is being raised to which power.

> **Investigate further:** Write out all four terms of $(x+2)^3$ and check they sum to 27 at $x = 1$. The "evaluate at $x = 1$" check catches most expansion slips in exams.

3. Row 5 of Pascal's triangle.

Answer: 1, 5, 10, 10, 5, 1

Method: Adjacent sums of row 4 $(1, 4, 6, 4, 1)$, or $\binom{5}{k}$ directly.

> **Investigate further:** Memorise rows 0–8 cold. Then look at the triangle mod 2 (odd entries only) — shade them for rows 0–16 and meet the Sierpiński triangle. D3 explains the pattern.

4. Coefficient of x^3y^2 in $(x+y)^5$.

Answer: $\binom{5}{2} = 10$

Method: Choose which 2 of the 5 brackets contribute y: $\binom{5}{2}$.

> **Investigate further:** Coefficients of x^3y^2 and x^2y^3 are equal — symmetry of $\binom{5}{2} = \binom{5}{3}$ made visible. In how many terms does $(x+y)^5$ expand? (6 — one per power split.)

5. Sum of all coefficients of $(1+x)^7$.

Answer: $2^7 = 128$

Method: Substitute $x = 1$: the expansion collapses to its coefficient sum.

> **Investigate further:** Substituting values into an identity is the sheet's master move. What does $x = 2$ give? (3^7 — the sum $\sum \binom{7}{k} 2^k$.) Every substitution is a new identity for free.

6. Constant term of $\left(x + \frac{1}{x}\right)^4$.

Answer: $\dbinom{4}{2} = 6$

Method: Powers cancel when the x's and $\frac{1}{x}$'s balance: 2 of each.

> **Investigate further:** Why does $\left(x + \frac{1}{x}\right)^5$ have *no* constant term? (Odd total degree — balance is impossible.) Parity kills entire terms before any computation.

7. Express $\dbinom{9}{4} + \dbinom{9}{5}$ as one coefficient.

Answer: $\dbinom{10}{5}$ (which is 252)

Method: Pascal's rule: adjacent entries sum to the entry below. No evaluation needed — the recognition is the answer.

> **Investigate further:** Prove it in one line with committees (a committee of 5 from 10 either contains the distinguished person or not) — you will need exactly this argument in C8.

8. $\dbinom{6}{0} - \dbinom{6}{1} + \dbinom{6}{2} - \cdots + \dbinom{6}{6}$.

Answer: 0

Method: Substitute $x = -1$ into $(1+x)^6$: $0^6 = 0$.

> **Investigate further:** Consequence: in any row, the even-index entries and odd-index entries have equal sums (each 2^{n-1} — B5 uses this). Pair with sheet 2's subset-parity bijection: same fact, two proofs.

9. True or false: coefficient of x^2 in $(1-x)^{10}$ is negative.

Answer: False — it is $+45$

Method: The x^2 term is $\binom{10}{2}(-x)^2 = 45x^2$: the square eats the sign. Signs alternate, and even powers are positive.

> **Investigate further:** TMUA-style sign traps live exactly here. Which coefficients of $(1-x)^{10}$ *are* negative? (The odd ones.) State the general sign rule for $(a-b)^n$ and never lose a mark to it again.

10. Largest coefficient in $(1+x)^4$.

Answer: $\binom{4}{2} = 6$

Method: Rows of Pascal's triangle rise to the middle and fall symmetrically.

> **Investigate further:** For even n there is one maximum; for odd n, two equal ones. Prove the rise-then-fall by examining the ratio $\binom{n}{k+1}/\binom{n}{k} = \frac{n-k}{k+1}$ — when does it drop below 1? (C2 uses this.)

Section B Manipulation Drills

($\leq$50 s each)

1. Coefficient of x^3 in $(2+3x)^5$.

Answer: $\binom{5}{3}2^2 3^3 = 10 \cdot 4 \cdot 27 = 1\,080$

Method: General term $\binom{5}{k}2^{5-k}(3x)^k$; set $k=3$. Both constants carry powers — forgetting 2^{5-k} is the classic slip.

> **Investigate further:** Check digit-cheaply: at $x=1$ the whole expansion is $5^5 = 3125$; your five coefficients must sum to it. Run the check — it takes 20 seconds and catches the missing-power error instantly.

2. Coefficient of x^5 in $(1+2x)^4(1+x)^3$.

Answer: 168

Method: Convolve the tails only: from $(1+2x)^4$ take coefficients $16, 32, 24$ (of x^4, x^3, x^2) and from $(1+x)^3$ take $3, 3, 1$ (of x, x^2, x^3): $16 \cdot 3 + 32 \cdot 3 + 24 \cdot 1 = 168$.

> **Investigate further:** Had both brackets shared the base $(1+x)$, you would merge first: $(1+x)^4(1+x)^3 = (1+x)^7$, coefficient $\binom{7}{5} = 21$ in five seconds. Check the bases before you convolve — and note that separate-then-convolve on the shared-base version proves $\sum_k \binom{4}{k}\binom{3}{5-k} = \binom{7}{5}$, a Vandermonde identity (D2).

3. Term independent of x in $\left(2x + \frac{1}{x^2}\right)^6$.

Answer: $\binom{6}{2}2^4 = 240$

Method: General term: $\binom{6}{k}(2x)^{6-k}x^{-2k}$ has degree $6-3k$. Zero degree forces $k=2$: $\binom{6}{2}2^4$.

> **Investigate further:** Degree bookkeeping first, arithmetic second. Which powers of x *can* occur here? ($6-3k$: only $6, 3, 0, -3, -6, -9, -12$ — gaps of 3.) Knowing the degree lattice tells you instantly that e.g. an x^2 term is impossible.

4. Coefficient of x^2 in $(1+x)^n$ is 66. Find n.

Answer: $n = 12$

Method: $\binom{n}{2} = 66 \Rightarrow n(n-1) = 132 = 12 \cdot 11$.

> **Investigate further:** Sheet 2's consecutive-product reflex again. Harder inverse: the coefficient of x^3 is 220 — find n ($n(n-1)(n-2) = 1320 = 12 \cdot 11 \cdot 10$). Estimation ($\sqrt[3]{1320} \approx 11$) then check.

5. Coefficient of x^6 in $\left(x^2 + \frac{1}{x}\right)^6$.

Answer: $\binom{6}{2} = 15$

Method: General term $\binom{6}{k}(x^2)^{6-k}x^{-k}$ has degree $12 - 3k$; degree 6 forces $k = 2$.

> **Investigate further:** The degree lattice here is $12, 9, 6, 3, 0, -3, -6$. Which is the constant term? ($k = 4$: $\binom{6}{4} = 15$ — equal to the x^6 coefficient. Coincidence, or a symmetry of this particular bracket? Substitute $x \to \frac{1}{x}$ and find out.)

6. Coefficient of x^2 in $(1 + 2x)^4(1 - x)^2$.

Answer: 9

Method: Convolve only what is needed: x^2 coefficient $= 24 \cdot 1 + 8 \cdot (-2) + 1 \cdot 1 = 9$, using $(1 + 2x)^4 = 1 + 8x + 24x^2 + \cdots$ and $(1 - x)^2 = 1 - 2x + x^2$.

> **Investigate further:** Never expand whole products for one coefficient — harvest each factor only up to the degree you need. Practise: coefficient of x^3 in the same product ($32 \cdot 1 + 24 \cdot (-2) + 8 \cdot 1 = -8$).

7. Evaluate 11^3 and 9^3 by binomial.

Answer: $11^3 = 1\,331; \quad 9^3 = 729$

Method: $(10 + 1)^3 = 1000 + 300 + 30 + 1$; $(10 - 1)^3 = 1000 - 300 + 30 - 1$. Pascal row 3 does the arithmetic.

> **Investigate further:** Notice 11^3's digits ARE row 3 of Pascal's triangle (1,3,3,1) — and $11^4 = 14641$ is row 4. When does the pattern break, and why? (Row 5 has a two-digit entry: carrying destroys the display.)

8. Coefficient of x^3 in $(1 + kx)^6$ is 160. Find k.

Answer: $k = 2$

Method: $\binom{6}{3}k^3 = 20k^3 = 160 \Rightarrow k^3 = 8$.

> **Investigate further:** Could a negative or fractional k ever be the intended answer in such questions? Solve $\binom{6}{3}k^3 = -540$ and $\binom{4}{2}k^2 = \frac{3}{2}$ to see both arise naturally in TMUA-style settings.

9. $\binom{n}{2} = \binom{n}{3}$. Find n.

Answer: $n = 5$

Method: Equal entries sit symmetrically: $\binom{n}{r} = \binom{n}{s}$ (with $r \neq s$) forces $r + s = n$. So $n = 2 + 3 = 5$.

> **Investigate further:** Solve without symmetry, by ratio: $\frac{\binom{n}{3}}{\binom{n}{2}} = \frac{n-2}{3} = 1$. The ratio form generalises to "for which k is $\binom{n}{k}$ maximal?" — close the loop with A10.

10. Coefficient of a^2b^2c in $(a + b + c)^5$. *(cf. TMUA 2020 P1 Q13)*

Answer: $\dfrac{5!}{2!\,2!\,1!} = 30$

Method: Multinomial: distribute the 5 brackets among the letters — exactly the BANANA formula from sheet 3 wearing algebra.

> **Investigate further:** Repeated-letter words, flag rows, multinomial coefficients: one formula, three sheets. How many *distinct monomials* does $(a+b+c)^5$ have? (Solutions of $i+j+k=5$: that is sheet 5's stars-and-bars, arriving tomorrow.)

Section C Substitution & Structure (≤90 s each)

1. $\dbinom{10}{0} + \dbinom{10}{2} + \cdots + \dbinom{10}{10}$.

Answer: $2^9 = 512$

Method: Substitute twice into $(1+x)^{10}$: $x=1$ gives 2^{10} (all terms), $x=-1$ gives 0 (alternating). Adding kills the odd-index terms and doubles the even ones: sum $= (2^{10}+0)/2 = 512$.

> **Investigate further:** The add/subtract-substitutions trick extracts any alternating pattern. Try extracting $\binom{10}{0} + \binom{10}{4} + \binom{10}{8}$ — what would you substitute? (Fourth roots of unity: a genuine BMO-level extension to sit with.)

2. Greatest coefficient in $(1+x)^8$.

Answer: $\dbinom{8}{4} = 70$

Method: Ratio test from A10's inv: $\frac{\binom{8}{k+1}}{\binom{8}{k}} = \frac{8-k}{k+1} > 1$ exactly while $k < 3.5$, so coefficients climb to $k=4$ then fall.

> **Investigate further:** Now find the greatest coefficient of $(1+2x)^8$ — the ratio becomes $\frac{2(8-k)}{k+1}$ and the peak moves right (to $k=5$: check it). Peaks shift towards the larger term; TMUA loves this.

3. Estimate 1.01^{10} from three terms.

Answer: $1 + 10(0.01) + 45(0.01)^2 = 1.1045$

Method: $(1+x)^{10}$ at $x = 0.01$; terms shrink so fast the tail is invisible at 4 d.p.

> **Investigate further:** Bound the error: the next term is $120(0.01)^3 = 1.2 \times 10^{-4}$... which *would* touch the 4th decimal place. Investigate: the true value is $1.10462\ldots$, so why did the estimate still round correctly? (The tail is positive but small enough. Always *bound*, not assume.)

4. In $(1+3x)^n$, coefficient of x^5 is three times that of x^4. Find n.

Answer: $n = 9$

Method: $\binom{n}{5}3^5 = 3 \cdot \binom{n}{4}3^4$. The powers of 3 cancel completely (3^5 on both sides), leaving $\binom{n}{5} = \binom{n}{4}$, so $n = 4+5 = 9$ by symmetry.

> **Investigate further:** The constants were a decoy — strip them before solving. Equal *adjacent* plain coefficients happen only for odd n at the centre (A10's two equal maxima). For a genuine constant

> effect, solve: in $(1+2x)^n$ the x^5 coefficient is *four* times the x^4 one; now the 2's don't cancel.

5. Coefficient of x^4 in $(1+x)^5(1-2x)^3$. *(cf. MAT 2023 Q1F)*

Answer: 25

Method: Convolution with signs, harvesting only degree-4 pairs: $5 \cdot 1 + 10 \cdot (-6) + 10 \cdot 12 + 5 \cdot (-8) = 5 - 60 + 120 - 40 = 25$. Lay the two coefficient lists side by side and zip them — do not expand the product.

> **Investigate further:** MAT's Q1 multiple-choice regularly hides one sign or one binomial power slip among its options. Recompute with the factors swapped in your zip to double-check — convolution is symmetric, so any asymmetry in your two answers locates the error.

6. $\displaystyle\sum_{k=0}^{5} k\binom{5}{k}$.

Answer: $5 \cdot 2^4 = 80$

Method: Committee–chair (sheet 2, B5): $k\binom{5}{k} = 5\binom{4}{k-1}$, and summing the right side gives $5 \cdot 2^4$. Alternatively differentiate $(1+x)^5$ and set $x = 1$.

> **Investigate further:** Two proofs from two different sheets — the counting one and the calculus one. Use either to evaluate $\sum k^2\binom{5}{k}$ (answer $5 \cdot 6 \cdot 2^3$ = check it: $k^2 = k(k-1) + k$). Identities stack.

7. Evaluate $\displaystyle\sum_{k=0}^{6} \binom{6}{k} 2^k$ by substitution.

Answer: $3^6 = 729$

Method: The sum is exactly $(1+x)^6$ at $x = 2$: $(1+2)^6 = 729$. Recognising a binomial expansion *in reverse* is the skill — the 2^k pattern names the substitution.

> **Investigate further:** Every value of x turns $(1+x)^n$ into an identity: $x = 10$ explains why row-n digits "read as" 11^n (B7), and $x = -\frac{1}{2}$ gives an alternating shrinking sum. Evaluate $\sum \binom{6}{k}(-1)^k 2^{6-k}$ by spotting *its* disguise ($(2-1)^6 = 1$).

8. Prove Pascal's rule by committee; build row 7 from row 6.

Answer: Row 7: $1, 7, 21, 35, 35, 21, 7, 1$

Method: *Proof.* A committee of r from n people either includes a distinguished person P or not. Including P: $\binom{n-1}{r-1}$ choices for the rest. Excluding P: $\binom{n-1}{r}$. These cases are disjoint and exhaustive, so $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$. $\square$ Adjacent sums of row 6 $(1, 6, 15, 20, 15, 6, 1)$ then give row 7.

> **Investigate further:** This one-sentence proof is a model of BMO1 writing: cases named, disjointness noted, conclusion drawn. Rewrite sheet 2 B9's Priya split as a formal proof in the same register — Section D now expects this standard.

Section D Challenge (SMC / BMO1 difficulty)

1. Prove $\displaystyle\binom{2n}{n}$ is even for all $n \geq 1$.

Answer: Proof below.

Method: *Proof.* By Pascal's rule, $\binom{2n}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n}$. By the symmetry $\binom{m}{r} = \binom{m}{m-r}$ applied with $m = 2n-1$: $\binom{2n-1}{n-1} = \binom{2n-1}{(2n-1)-(n-1)} = \binom{2n-1}{n}$. So $\binom{2n}{n}$ is the sum of two *equal* integers, hence even. □

Method: *Alternative (bijective) proof.* Pair each n-subset S of $\{1, \ldots, 2n\}$ with its complement S^c. Since $|S| = n = |S^c|$ and $S \neq S^c$ always, the n-subsets split into disjoint pairs, so their number is even. □

> **Investigate further:** Two complete written proofs — study how each names its key step. Sharper question: what is the largest power of 2 dividing $\binom{2n}{n}$? Compute it for $n = 1, \ldots, 8$ and conjecture (Kummer's theorem: the number of carries when adding $n + n$ in binary).

2. Show $\displaystyle\sum_{k=0}^{10} \binom{10}{k}^2 = \binom{20}{10}$.

Answer: Both count $\binom{20}{10} = 184\,756$.

Method: *Proof.* Consider choosing 10 people from a room of 10 women and 10 men: $\binom{20}{10}$ ways. Alternatively, classify each choice by the number k of women chosen: $\binom{10}{k}$ ways for the women and $\binom{10}{10-k}$ for the men, and $\binom{10}{10-k} = \binom{10}{k}$ by symmetry. Summing the disjoint classes: $\sum_k \binom{10}{k}^2$. Two counts of one set are equal. □

> **Investigate further:** This is Vandermonde's identity $\sum_k \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$ at $m = n = r = 10$ — sheet 2's C5 inv, now proved properly. Also prove it by extracting the x^{10} coefficient from $(1+x)^{10}(1+x)^{10}$: the algebra proof and the committee proof should feel like translations of each other.

3. Prove exactly four of the $\displaystyle\binom{12}{k}$ are odd. *(cf. TMUA 2023 P1 Q6)*

Answer: The odd ones are exactly $k = 0, 4, 8, 12$.

Method: *Proof.* All congruences are mod 2. First, $(1+x)^2 = 1 + 2x + x^2 \equiv 1 + x^2$, and squaring again, $(1+x)^4 \equiv (1+x^2)^2 \equiv 1 + x^4$. Hence $(1+x)^{12} = \left((1+x)^4\right)^3 \equiv (1+x^4)^3 = 1 + 3x^4 + 3x^8 + x^{12} \equiv 1 + x^4 + x^8 + x^{12}$. Comparing coefficients of x^k on both sides: $\binom{12}{k}$ is odd precisely when $k \in \{0, 4, 8, 12\}$ — four values. □

> **Investigate further:** General law (Lucas): $\binom{n}{k}$ is odd iff every binary digit of k is $\leq$ the corresponding digit of n. Here $n = 12 = 1100_2$, so $k \in \{0000, 0100, 1000, 1100\}$ — the same four. Verify Lucas on row 10 (1010_2: which k?), and connect to A3's Sierpiński picture.

4. Show $\displaystyle\binom{2}{2} + \binom{3}{2} + \cdots + \binom{9}{2} = \binom{10}{3}$.

Answer: $\displaystyle\binom{10}{3} = 120$

Method: *Proof (classify by largest element).* Count the 3-subsets of $\{1, 2, \ldots, 10\}$: there are $\binom{10}{3}$. Classify each subset by its largest element $m+1$ (so $m+1$ ranges over $3, \ldots, 10$): the other two elements are then any 2-subset of $\{1, \ldots, m\}$, giving $\binom{m}{2}$ subsets in that class. The classes are disjoint and exhaustive, so $\sum_{m=2}^{9} \binom{m}{2} = \binom{10}{3}$. □

> **Investigate further:** This is the *hockey stick identity*: any diagonal run in Pascal's triangle sums to the entry below-right of its end. Prove it a second way by iterating Pascal's rule. Then use it to re-derive $1 + 2 + \cdots + n = \binom{n+1}{2}$ and $\sum m^2$ (write $m^2 = 2\binom{m}{2} + \binom{m}{1}$).

5. p prime, $0 < k < p$: prove $p \mid \dbinom{p}{k}$.

Answer: Proof below.

Method: *Proof.* Write $k!\binom{p}{k} = p(p-1)\cdots(p-k+1)$. The right side is visibly divisible by p, so $p \mid k!\binom{p}{k}$. Since p is prime and every factor of $k! = 1 \cdot 2 \cdots k$ is less than p, we have $p \nmid k!$, so p and $k!$ are coprime. A prime dividing a product must divide one of the factors; as it cannot divide $k!$, it divides $\binom{p}{k}$. $\square$ (Note where primality is used — *twice*.)

> **Investigate further:** Consequence: $(1 + x)^p \equiv 1 + x^p \pmod p$ — the "freshman's dream" is a theorem mod p. Feed $x = 1$ into it to get $2^p \equiv 2 \pmod p$: you have proved Fermat's little theorem for base 2. Test the failure for composites: is $\binom{4}{2}$ divisible by 4? This lemma returns as a workhorse in the number-theory pillar.

Top 5 Patterns Today

1. **The general term.** Write $\binom{n}{k}a^{n-k}b^k$ *with the constants inside*, solve the degree equation for k, then compute. Never expand what you can index.
2. **Substitute values into identities.** $x = 1$: coefficient sum. $x = -1$: alternating sum. Add/subtract the two: even/odd extraction. Every substitution is a theorem.
3. **Pascal's rule = in-or-out.** $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ is the committee split from sheet 2, and it generates the whole triangle.
4. **Count one thing two ways.** Vandermonde, hockey stick, committee–chair: pick the right scenario and the identity proves itself in a sentence.
5. **Binomials mod small primes.** $(1+x)^2 \equiv 1+x^2 \pmod 2$ propagates to $(1+x)^{2^m} \equiv 1+x^{2^m}$: parity patterns of whole rows fall out (Lucas/Sierpiński).

Common Traps to Avoid

1. **Dropping the constant's power.** In $(2 + 3x)^5$ the 2^{5-k} is compulsory. Coefficient-sum check at $x = 1$ catches this in seconds.
2. **Sign slips with $(a - b)^n$.** Keep $(-b)^k$ intact inside the general term; decide the sign from the parity of k, never by feel.
3. **Expanding both factors fully for one coefficient.** Harvest each factor only to the degree needed and zip. Full expansion is slow and error-prone.
4. **Assuming symmetric-looking sums need algebra.** $\sum \binom{10}{k}^2$ falls to a story about choosing 10 people. Reach for the committee before the formula sheet.
5. **Using primality nowhere.** If your proof of D5 never invokes that p is prime, it proves the false statement $4 \mid \binom{4}{2}$. Locate where every hypothesis earns its keep.

"An identity is a story told twice — once for each side of the equals sign."

Found a mistake or want to contribute a sheet?
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