Daily Combinatorics Drill #3 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Blocks & Gaps · Circular Arrangements · Repeated Letters · Forced-Order Tricks.

Section A Rapid Recognition ($\leq$15 s each)

1. In how many ways can 5 people stand in a row?

Answer: $5! = 120$

Method: Distinct objects in a row: factorial. Instant.

> **Investigate further:** Sheet 1's opening question returns as pure recall — that is the point of Section A. If this took more than three seconds, redo sheet 1's A section tomorrow.

2. How many arrangements of the letters of LEMON?

Answer: $5! = 120$

Method: Five *distinct* letters — same question as A1 in costume.

> **Investigate further:** Spot the isomorphism reflex: people in rows, distinct letters, distinct books are the same problem. The next question breaks the spell — watch what changes.

3. How many arrangements of the letters of BANANA?

Answer: $\dfrac{6!}{3!\,2!} = 60$

Method: Six letters: A$\times$3, N$\times$2, B$\times$1. Divide 6! by the repeats: $\frac{720}{6\cdot 2}$.

> **Investigate further:** Why division works: the 3 A's can be permuted in 3! ways without changing the word, so each distinct word was counted exactly $3!\,2!$ times among the 6!. This "quotient out the symmetry" idea is the same one that will give circular arrangements in A4.

4. Four people around a round table (rotations the same)?

Answer: $(4-1)! = 6$

Method: Fix one person to kill the rotational symmetry; arrange the remaining 3 freely.

> **Investigate further:** Alternative view: $\frac{4!}{4}$ — each circular arrangement is counted 4 times among row arrangements, once per rotation. Check both give 6. When would you also divide by 2? (When the table can be "flipped" — necklaces, handshake circles.)

5. Six in a row, Ali first and Zara last?

Answer: $4! = 24$

Method: Two positions forced, four people free.

> **Investigate further:** Forced positions contribute factor 1 — cousins of sheet 2's forced aces. Variant: Ali *or* Zara first (and no other constraint): $2 \cdot 5! = 240$. Why can you not just double 24?

6. Three identical red flags and two identical blue flags in a row?

Answer: $\binom{5}{2} = 10$

Method: Only the *positions* of the blue flags matter: choose 2 places from 5. Equivalently $\frac{5!}{3!2!}$.

> **Investigate further:** Repeated-letter arrangements and binomial coefficients are the same formula meeting in the middle: $\frac{5!}{3!2!} = \binom{5}{2}$. Every binary string question (sheet 2, B6) is a flag question. What is the three-colour version? (The multinomial coefficient — sheet 4.)

7. Arrangements of NOON?

Answer: $\dfrac{4!}{2!\,2!} = 6$

Method: Four letters, two pairs: $\frac{24}{4}$.

> **Investigate further:** List all six to feel the formula: NOON, NONO, NNOO, OONN, ONON, ONNO. Any word whose repeat signature is $(2, 2)$ has 6 arrangements — signature, not spelling, decides the count.

8. Six people around a round table (rotations the same)?

Answer: $(6 - 1)! = 120$

Method: Fix one person, arrange 5.

> **Investigate further:** A dinner-party host claims seating 6 guests differently every night is possible for four months. Check: 120 nights $\approx$ four months. For how long could 8 guests last? (Nearly 14 years.)

9. True or false: 8 people round a table = 7 people in a row?

Answer: True — both are $7! = 5040$

Method: Circular $n = (n - 1)! =$ row arrangements of $n - 1$. Fixing one diner *is* reducing to a row of the rest.

> **Investigate further:** TMUA-style equivalence spotting. Decide true/false: "$2n$ people in a circle = n people in a row, for some $n > 1$." ($(2n - 1)! = n!$ forces $2n - 1 = n$, impossible for $n > 1$ — false.)

10. Arrangements of TATTY?

Answer: $\dfrac{5!}{3!} = 20$

Method: Five letters with T$\times$3: divide by $3!$.

> **Investigate further:** Rank the words BANANA (60), TATTY (20), LEVEL (30), LEMON (120) by arrangement count *before* computing, using only the repeat structure. Fluency at reading the "signature" $(3, 2, 1)$ vs $(3, 1, 1)$ vs $(2, 2, 1)$ is the skill.

Section B Manipulation Drills ($\leq$50 s each)

1. Six in a row, Priya and Quinn together?

Answer: $2 \cdot 5! = 240$

Method: Glue the pair (block has 2 internal orders), arrange 5 objects.

> **Investigate further:** Sheet 1 met this in Section C; now it is a B-level reflex — the spiral is deliberate. Speed target: write "$2 \cdot 5!$" within five seconds of finishing reading.

2. Six in a row, Priya and Quinn *not* together?

Answer: $720 - 240 = 480$

Method: Complement of B1.

> **Investigate further:** Reprove via gaps: seat the other 4 (4!), then place P and Q in different gaps of the 5 available: $4! \cdot 5 \cdot 4 = 480$. The gap method scales to "no two of *three* people adjacent" where complements get messy — see D5.

3. Arrangements of MISSISSIPPI?

Answer: $\dfrac{11!}{4!\,4!\,2!} = 34\,650$

Method: 11 letters: I×4, S×4, P×2, M×1. One division, no listing.

> **Investigate further:** Compute it calculator-free the clean way: $\frac{11!}{4!4!2!} = \binom{11}{4}\binom{7}{4}\binom{3}{2} = 330 \cdot 35 \cdot 3$. Choosing positions letter-by-letter both proves the formula and organises the arithmetic.

4. Five boys, three girls in a row, no two girls adjacent?

Answer: $5! \cdot (6 \cdot 5 \cdot 4) = 120 \cdot 120 = 14\,400$

Method: Gap method: arrange the boys (5!), creating 6 gaps (including the two ends); place the three distinct girls in three *different* gaps, in order: $6 \cdot 5 \cdot 4$.

> **Investigate further:** Why does gluing/complement fail here? "No two of three adjacent" has too many overlapping bad events. The gap method sidesteps inclusion–exclusion entirely — but inclusion–exclusion (sheet 5) will handle what gaps cannot.

5. Four men and four women in a row, alternating?

Answer: $2 \cdot 4! \cdot 4! = 1\,152$

Method: Two gender patterns (MWMW... or WMWM...), then the men fill their 4 slots in 4! ways and the women theirs in 4!.

> **Investigate further:** With 5 men and 4 women only one pattern survives (MWMWMWMWM): $5! \cdot 4! = 2880$, no factor 2. The factor-of-2 question is always: does the *pattern* have a choice, or is it forced?

6. Six around a round table, two particular people together?

Answer: $2 \cdot (5-1)! = 48$

Method: Glue the pair into a block: 5 objects round a table $= (5-1)!$, times 2 internal orders.

> **Investigate further:** Row version was $2 \cdot 5! = 240$; circle version is $2 \cdot 4! = 48$. The ratio is 5 — exactly the number of rotations quotiented out. Check this ratio logic predicts the answer for 8 people with a glued pair.

7. Six-digit strings from $1, 1, 2, 2, 3, 3$?

Answer: $\dfrac{6!}{2!\,2!\,2!} = 90$

Method: Three pairs: divide 720 by 2^3.

> **Investigate further:** Of these 90, how many are palindromes? (A palindrome is fixed by its first three digits, which must use each of 1, 2, 3 exactly once: $3! = 6$.) Symmetry constraints slash repeated-letter counts — try "how many read the same upside down" style variants.

8. Arrangements of LEVEL?

Answer: $\dfrac{5!}{2!\,2!} = 30$

Method: L×2, E×2, V: divide 120 by 4.

> **Investigate further:** How many of the 30 are palindromes? (Middle letter forced to V; first two letters L, E in some order determine the rest: 2.) Notice palindrome counting always halves the alphabet workload: only half the word is free.

9. Seven books, trilogy in volume order (not necessarily adjacent)?

Answer: $\dfrac{7!}{3!} = 840$

Method: Forced-order trick: of the 3! orderings of the trilogy within any arrangement, exactly one is correct — so divide 7! by 3!. No gluing: they need not be adjacent.

> **Investigate further:** Compare the three trilogy conditions: adjacent-any-order ($3! \cdot 5! = 720$), adjacent-in-order ($5! = 120$), in-order-anywhere (840). Rank them by restrictiveness *before* computing next time. The divide-by-order trick is sheet 1's "strictly increasing digits" insight, now industrialised.

10. Round-table seatings of n people $= 720$. Find n.

Answer: $n = 7$

Method: $(n-1)! = 720 = 6!$, so $n = 7$.

> **Investigate further:** Factorials are so sparse that "$= 720$" identifies 6! instantly — recall sheet 1's ladder. Solve also $(n-1)! \cdot 2 = 240$ (a glued-pair circle count): $n = 6$.

Section C Substitution & Structure (≤90 s each)

1. Eight in a row: three friends consecutive, two others (non-friends) not adjacent?

Answer: $3! \cdot 6! - 3! \cdot 2! \cdot 5! = 4\,320 - 1\,440 = 2\,880$

Method: Layer the constraints (sheet's D2 pattern, arriving early). Glue the trio: $3! \cdot 6! = 4\,320$ arrangements. From these subtract those where the two enemies are also adjacent — glue them too: $3! \cdot 2! \cdot 5! = 1\,440$. Answer $2\,880$.

> **Investigate further:** Generalise the single-constraint count: k specified people consecutive among n gives $k!\,(n-k+1)!$ — check it against B1. Then redo this question with the gap method instead of the complement and confirm agreement.

2. Same eight, no refusals, friends consecutive *and* in height order?

Answer: $6! = 720$

Method: The block's internal order is now forced: glue with factor 1, then 6 objects in a row.

> **Investigate further:** "Consecutive AND ordered" = glue with factor 1; "ordered but scattered" = divide by 3! (B9). Keep the two tricks distinct in your head — mixing them up is a classic exam slip.

3. Arrangements of DIVIDED without all three D's together?

Answer: $\dfrac{7!}{3!\,2!} - \dfrac{5!}{2!} = 420 - 60 = 360$

Method: Total arrangements: $\frac{7!}{3!2!} = 420$. All-D's-together: glue [DDD] (identical letters, factor 1) and arrange it with I, I, V, E: $\frac{5!}{2!} = 60$. Subtract.

> **Investigate further:** Careful distinction: the complement of "all three together" still allows *two* D's adjacent. Count the arrangements where no two D's are adjacent at all (gap method on I, I, V, E: $\frac{4!}{2!} \cdot \binom{5}{3} = 120$) and check $120 < 360$ makes sense.

4. Four men, four women alternating around a round table?

Answer: $3! \cdot 4! = 144$

Method: Seat the men first: circular, so $(4-1)! = 6$. Their seating fixes 4 alternating gaps, into which the women go in $4!$ ways. (No extra factor 2: in a circle the two "patterns" are rotations of each other.)

> **Investigate further:** Row version (B5) had the pattern factor 2; the circle ate it. Trace exactly where: rotating an MWMW… row by one seat turns it into WMWM… — rotations already identify the two patterns. Symmetries must be counted once, not twice.

5. Arrangements of CIRCLE with the two C's apart?

Answer: $\dfrac{6!}{2!} - 5! = 360 - 120 = 240$

Method: Total (with C repeated): 360. Together: glue CC (identical, no internal factor): $5! = 120$. Subtract.

> **Investigate further:** Redo via gaps: arrange I, R, L, E ($4!$), choose 2 of the 5 gaps for the identical C's ($\binom{5}{2}$): $24 \cdot 10 = 240$. ✓ Complement and gaps agree — and for *identical* letters the gap count uses $\binom{}{}$, not a falling product. Spot why.

6. Five-digit even numbers over $40\,000$ using $1, 2, 3, 4, 5$ once each?

Answer: 18

Method: Two interacting constraints (units even, leading digit 4 or 5) — case on the units digit, because it consumes a candidate leading digit or not. *Units 2:* leading digit from $\{4, 5\}$, rest free: $2 \cdot 3! = 12$. *Units 4:* leading digit must be 5, rest free: $3! = 6$. Total 18.

> **Investigate further:** Why case on units rather than the leading digit? Try it the other way and watch the option-counts stop being uniform (sheet 1's C1 lesson). Extension: how many are divisible by 4? (Endings 12, 24, 32, 52; now impose the $> 40\,000$ condition on each case.)

7. Three maths, two physics, two chemistry books, subjects together?

Answer: $3! \cdot (3! \cdot 2! \cdot 2!) = 6 \cdot 24 = 144$

Method: Arrange the three subject blocks ($3!$), then the books inside each block ($3!, 2!, 2!$).

> **Investigate further:** Blocks-of-blocks compose: shelves of subjects of books of chapters... each level multiplies independently. Now add the constraint "maths must be left of chemistry" — which factor changes, and to what? (The 3! becomes 3: divide by the 2! orderings of the M–C pair.)

8. Eight around a round table, two people refusing to be adjacent?

Answer: $7! - 2 \cdot 6! = 5\,040 - 1\,440 = 3\,600$

Method: All seatings ($7!$) minus glued-pair seatings ($2 \cdot 6!$).

> **Investigate further:** Gap version: seat the 6 others round the table ($5!$), then put the two enemies into 2 of the 6 gaps: $5! \cdot 6 \cdot 5 = 3600$. ✓ Both routes again — build the habit of solving restricted seatings twice until they always agree first time.

Section D Challenge (SMC difficulty)

1. Seven-digit numbers from 1–7: odds increasing, evens decreasing?

Answer: $\binom{7}{4} = 35$

Method: Choose which 4 of the 7 positions hold the odd digits: $\binom{7}{4} = 35$. Everything else is forced — the odds $1, 3, 5, 7$ must fill their positions in increasing order (one way), and the evens $6, 4, 2$ fill theirs in decreasing order (one way). A count that looks like it needs 7! cases collapses to a single choice.

> **Investigate further:** Equivalently by double forced-order division: $\frac{7!}{4!\,3!} = 35$ — divide by the orderings of the odds AND of the evens. With only the odds constrained the count is $\frac{7!}{4!} = 210$; check both against the choose-positions view. This "order is forced, so only positions matter" insight is the SMC's favourite disguise for $\binom{n}{k}$.

2. Ten children in a circle: Mia–Noah adjacent, Omar–Priya not?

Answer: $2 \cdot 8! - 4 \cdot 7! = 80\,640 - 20\,160 = 60\,480$

Method: Layer the constraints. Glue Mia–Noah: $2 \cdot (9 - 1)! = 80\,640$ circles. From these subtract the ones where Omar–Priya are also adjacent (glue both pairs): $2 \cdot 2 \cdot (8 - 1)! = 20\,160$. Answer: $60\,480$.

> **Investigate further:** The pattern "force the easy constraint, complement the hard one" handles most two-constraint arrangement problems. Try three pairs: A–B adjacent, C–D adjacent, E–F *not*

> adjacent, 10 people in a circle. (Same layering, one more glue.)

3. Arrangements of AABBCC with no two identical letters adjacent?

Answer: 30

Method: Inclusion–exclusion on the "bad pair" events, computed cleanly by gluing. Total: $\frac{6!}{2!2!2!} = 90$. Glue AA: $\frac{5!}{2!2!} = 30$, same for BB, CC. Glue two pairs: $\frac{4!}{2!} = 12$ each (3 ways). Glue all three: $3! = 6$. Count with no glued pair $= 90 - 3 \cdot 30 + 3 \cdot 12 - 6 = 30$.

> **Investigate further:** This is inclusion–exclusion arriving a sheet early — sheet 5 makes it systematic. For now, verify the answer differently: by symmetry all 3! letter-labelings are equivalent, so just count patterns like ABCABC, ABCACB, … and multiply. How many *patterns* are there? ($30/3! = 5$.)

4. Eight in a row, exactly two of the three sisters adjacent?

Answer: $8! - 14\,400 - 4\,320 = 21\,600$

Method: Three disjoint outcomes partition all $8! = 40\,320$ arrangements: *no* adjacent sister-pair, *exactly one* adjacent sister-pair, and *two* adjacent sister-pairs (which is precisely "all three consecutive"). *None adjacent* (gap method): $5! \cdot 6 \cdot 5 \cdot 4 = 14\,400$. *All three consecutive*: $3! \cdot 6! = 4\,320$. "Exactly two sisters adjacent" is the middle category, so it equals $40\,320 - 14\,400 - 4\,320 = 21\,600$.

> **Investigate further:** Check the borderline case that makes this work: in a block of three sisters, the outer two are NOT adjacent to each other — adjacency is about neighbouring seats, not membership of a block. Redo the problem for "at least two adjacent" ($40\,320 - 14\,400 = 25\,920$) and confirm the three-way split is consistent.

5. Eight knights round a table: three feuders pairwise non-adjacent, King and Queen together?

Answer: $3! \cdot 2 \cdot (4 \cdot 3 \cdot 2) = 288$

Method: Layer: glue first, then gaps. Glue the King–Queen pair (2 internal orders); the peaceful side is now 4 units (the KQ block plus 3 other knights). Seat those 4 units round the table: $(4-1)! = 6$ circular orders, creating exactly 4 gaps. Place the three feuding knights into three different gaps (one each — sharing a gap would make two feuders adjacent): $4 \cdot 3 \cdot 2 = 24$. Total $6 \cdot 2 \cdot 24 = 288$.

> **Investigate further:** Check the order of operations matters: try placing the feuders first and gluing second, and watch the gap count become ill-defined — constraints that *create* structure (glue) must precede constraints that *consume* it (gaps). Then generalise the no-royals version: n knights, k feuding, count $= (n-k-1)! \cdot \frac{(n-k)!}{(n-2k)!}$; test it at $n = 8$, $k = 3$ (1440) against a hand count of a small case like $n = 5$, $k = 2$.

Top 5 Patterns Today

1. **Glue blocks.** "Must be together" ⇒ treat as one object, multiply by internal orders (which is 1 if the glued items are identical or their order is forced).

2. **Count gaps.** "No two adjacent" ⇒ seat the unrestricted objects, then place the restricted ones into distinct gaps. Rows of m give $m + 1$ gaps; circles of m give m gaps.

3. **Divide by forced order.** If k objects must appear in one specified relative order (not necessarily adjacent), count freely and divide by $k!$.

4. **Fix a seat in circles.** Rotational symmetry costs one factor of n: circular $n = (n-1)!$. Ask

separately whether reflections are also “the same”.

5. **Layer constraints.** Force the “must” constraints by gluing first, then handle the “must not” by complement or gaps inside that restricted world.

Common Traps to Avoid

1. **Multiplying by internal orders of identical letters.** Gluing the two E’s of SPEED is a factor 1, not 2. Only *distinguishable* contents of a block get a factorial.

2. **Row gaps vs circle gaps.** Five people create 6 gaps in a row but only 5 in a circle. Using the row count at a round table silently overcounts.

3. **Double-counting the alternating patterns in a circle.** MWMW… and WMWM… are rotations of each other at a round table — the row-version factor 2 must be dropped (C4).

4. **Confusing “in order” with “adjacent”.** Volume order scattered along the shelf divides by 3!; volumes boxed together glues a block. Different tricks, different answers (B9 vs C1/C2).

5. **Treating “exactly one adjacent pair” as total minus none-adjacent.** That difference counts *at least one* pair. The all-together case must be carved out separately (D4).

“Glue what must be together, seat what must be apart last — the gaps will tell you how much room rebellion really has.”

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths