Daily Combinatorics Drill #2 — Answers & Investigations

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Section	**Topic**	**Questions**	**Target time**
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Combinations $\binom{n}{r}$ · Choosing vs Arranging · Complementary Counting · Committee Constructions.

Section A Rapid Recognition

($\leq$15 s each)

1. Evaluate: $\dbinom{7}{2}$.

Answer: 21

Method: $\dbinom{7}{2} = \dfrac{7 \cdot 6}{2} = 21$. For $\binom{n}{2}$, always: half of $n(n-1)$.

> **Investigate further:** $\binom{n}{2}$ is the $(n-1)$th triangular number. Why must the product of two consecutive integers always be even, making $\binom{n}{2}$ an integer? This integrality question generalises to all $\binom{n}{r}$ — see B-section of sheet 6.

2. Evaluate: $\dbinom{8}{3}$.

Answer: 56

Method: $\dbinom{8}{3} = \dfrac{8 \cdot 7 \cdot 6}{3!} = \dfrac{336}{6} = 56$. Cancel before multiplying: $\frac{8 \cdot 7 \cdot 6}{6} = 8 \cdot 7$.

> **Investigate further:** Note $\binom{8}{3} = 56 = \binom{8}{5}$. Prove the symmetry $\binom{n}{r} = \binom{n}{n-r}$ *without* algebra: choosing which 3 objects to take is the same decision as choosing which 5 to leave behind.

3. Evaluate: $\dbinom{9}{7}$.

Answer: 36

Method: Use symmetry first: $\dbinom{9}{7} = \dbinom{9}{2} = \dfrac{9 \cdot 8}{2} = 36$. Never compute a large lower index directly.

> **Investigate further:** How large can the saving get? Compare the work in computing $\binom{100}{98}$ directly against $\binom{100}{2}$. Under exam time pressure, the symmetry flip is the single most-used binomial reflex.

4. How many committees of 3 can be chosen from 6 people?

Answer: $\dbinom{6}{3} = 20$

Method: A committee is unordered: $\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{6} = 20$.

> **Investigate further:** Contrast with sheet 1's medal question: 3 *distinct* medals from 8 runners was $8 \cdot 7 \cdot 6$. Divide-by-$r!$ is exactly the bridge between the two: $\binom{n}{r} = \frac{^nP_r}{r!}$. When does a question secretly order the chosen objects?

5. Solve for n: $\binom{n}{2} = 45$.

Answer: $n = 10$

Method: $\frac{n(n-1)}{2} = 45 \Rightarrow n(n-1) = 90 = 10 \cdot 9 \Rightarrow n = 10$. Spot consecutive-integer products instantly.

> **Investigate further:** Which totals T make $\binom{n}{2} = T$ solvable? Exactly the triangular numbers $1, 3, 6, 10, 15, \ldots$ Investigate: can a number be triangular in two different ways?

6. How many subsets of a 5-element set have at least 4 elements?

Answer: $\binom{5}{4} + \binom{5}{5} = 5 + 1 = 6$

Method: "At least 4" means sizes 4 or 5. Two small cases — add them.

> **Investigate further:** "At least" with *few* qualifying cases: add directly. "At least" with *many* qualifying cases: subtract the complement. Recount this both ways ($32 - \binom{5}{0} - \binom{5}{1} - \binom{5}{2} - \binom{5}{3} = 6$) and note which was faster — then decide the crossover rule for yourself.

7. Evaluate: $\binom{6}{0} + \binom{6}{1} + \cdots + \binom{6}{6}$.

Answer: $2^6 = 64$

Method: Row sum of Pascal's triangle: every subset of a 6-set has *some* size, so the sizes partition the 2^6 subsets.

> **Investigate further:** This double count (subsets by size vs. in/out choices) is your first serious "count the same thing two ways" argument — the headline tool of sheet 6. Also evaluate the alternating version $\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \cdots$ (answer 0) and explain it by pairing subsets.

8. How many ways can 2 letters be chosen from {A, B, C, D, E}?

Answer: $\binom{5}{2} = 10$

Method: Unordered pairs from 5: $\frac{5 \cdot 4}{2} = 10$.

> **Investigate further:** List all 10 pairs quickly in lexicographic order (AB, AC, . . . , DE). Systematic listing is worth practising: it is the verification tool of last resort when a formula feels doubtful in an exam.

9. True or false: $\binom{100}{98} = 4950$.

Answer: True

Method: $\dbinom{100}{98} = \dbinom{100}{2} = \dfrac{100 \cdot 99}{2} = 4950.$

> **Investigate further:** TMUA-style statement checks reward one clean disproof or one clean identity. Decide true/false: $\binom{n}{2} + \binom{n+1}{2} = n^2$. (True — prove it, then interpret it as two staircase halves of a square.)

10. A lottery ticket selects 4 numbers from 1–10 (order irrelevant). How many different tickets are there?

Answer: $\dbinom{10}{4} = 210$

Method: Pure choice, no order: $\dfrac{10 \cdot 9 \cdot 8 \cdot 7}{24} = 210.$

> **Investigate further:** The UK Lotto uses 6 numbers from 59: $\binom{59}{6} = 45\,057\,474$. Estimate it mentally as $\frac{59^6}{720}$ to one significant figure and compare — order-of-magnitude fluency beats exact arithmetic here.

Section B Manipulation Drills

(≤50 s each)

1. Solve for n: $\dbinom{n}{3} = 35.$

Answer: $n = 7$

Method: $n(n-1)(n-2) = 35 \cdot 6 = 210 = 7 \cdot 6 \cdot 5$. Three consecutive integers — read off $n = 7$.

> **Investigate further:** Estimate first: $n(n-1)(n-2) \approx n^3$, so $n \approx \sqrt[3]{210} \approx 6$; test 6 and 7. The estimate-then-check loop solves every $\binom{n}{r} = T$ equation faster than algebra.

2. From 8 men and 6 women, how many committees of 2 men and 2 women can be formed?

Answer: $\dbinom{8}{2}\dbinom{6}{2} = 28 \cdot 15 = 420$

Method: Independent choices multiply: choose the men, then the women.

> **Investigate further:** Why is it *not* $\binom{14}{4}$ restricted afterwards? Compute how many of the $\binom{14}{4} = 1001$ unrestricted committees have the 2–2 split, and check you recover 420. Structure-first beats filter-later.

3. Twelve people at a party all shake hands with each other exactly once. How many handshakes take place?

Answer: $\dbinom{12}{2} = 66$

Method: A handshake *is* an unordered pair of people.

> **Investigate further:** Recount by degrees: each of 12 people shakes 11 hands, and $12 \cdot 11$ counts every handshake twice — so 66. This degree-sum argument is the Handshake Lemma; it returns in sheet 6 and in graph theory proper.

4. Ten points, no three collinear: how many segments, and how many triangles?

Answer: $\binom{10}{2} = 45$ segments; $\binom{10}{3} = 120$ triangles

Method: A segment is a pair of points; a triangle is a triple. The no-three-collinear condition guarantees every triple genuinely forms a triangle.

> **Investigate further:** Now let exactly 4 of the 10 points be collinear. How many triangles survive? ($120 - \binom{4}{3} = 116$.) Degenerate cases are subtracted, not case-split — a complementary-counting reflex.

5. Team of 5 from 11, then a captain from the team. Count both ways.

Answer: $\binom{11}{5} \cdot 5 = 2310 = 11 \cdot \binom{10}{4}$

Method: Team first: $462 \cdot 5$. Captain first: 11 choices, then 4 teammates from the remaining 10: $11 \cdot 210$. Both give 2310.

> **Investigate further:** You have just proved the identity $r\binom{n}{r} = n\binom{n-1}{r-1}$ by counting one committee-with-chair two ways. Write the algebraic proof too, and keep both — "committee–chair" arguments prove many binomial identities without algebra.

6. How many binary strings of length 8 contain exactly three 1s?

Answer: $\binom{8}{3} = 56$

Method: A string is determined by *which* 3 of the 8 positions hold the 1s.

> **Investigate further:** "Choose the positions" converts string problems into subset problems. How many length-8 strings have more 1s than 0s? (Sum $\binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8} = 93$.) Could symmetry have halved the work? Careful — the middle term $\binom{8}{4}$ belongs to neither side.

7. From the 5 vowels and 21 consonants, how many 3-letter *sets* have exactly one vowel and two consonants?

Answer: $\binom{5}{1}\binom{21}{2} = 5 \cdot 210 = 1\,050$

Method: Choose the vowel, then the pair of consonants — independent choices multiply.

> **Investigate further:** Now count 3-letter *strings* (ordered, no repeats) with exactly one vowel: multiply by the 3 positions for the vowel and 2! orders of the consonants — or by 3! on the whole set. Ordered = unordered × arrangements, always.

8. Solve for n: $\binom{n}{2} + \binom{n}{1} = 21$.

Answer: $n = 6$

Method: $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2} = 21 \Rightarrow n(n+1) = 42 = 6 \cdot 7 \Rightarrow n = 6$.

> **Investigate further:** You used Pascal's rule backwards: $\binom{n}{2} + \binom{n}{1} = \binom{n+1}{2}$. State Pascal's rule

> $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ and prove it by asking whether a distinguished element is in the chosen set — sheet 4 builds the whole triangle from this.

9. Committees of 4 from 9 people: how many include Priya? Exclude her?

> **Answer:** Include: $\binom{8}{3} = 56$; exclude: $\binom{8}{4} = 70$

Method: Include her: she is in, choose 3 companions from 8. Exclude her: choose all 4 from the other 8.

> **Investigate further:** $56 + 70 = 126 = \binom{9}{4}$ — every committee either contains Priya or doesn't. This in/out split *is* Pascal's rule with faces on it. Use the same split to compute $\binom{10}{5}$ from row 9.

10. Routes from $(0, 0)$ to $(4, 3)$ stepping only right or up?

> **Answer:** $\binom{7}{3} = 35$

Method: Every route is a string of 4 R's and 3 U's: choose which 3 of the 7 steps are U.

> **Investigate further:** Lattice paths are strings are subsets — one idea, three costumes. How many routes pass through the point $(2, 1)$? (Multiply the two stages: $\binom{3}{1}\binom{4}{2} = 18$.) How many *avoid* it?

Section C Substitution & Structure ($\leq$90 s each)

1. Committee of 5 from 7 women and 6 men, at least 4 women.

> **Answer:** $\binom{7}{4}\binom{6}{1} + \binom{7}{5} = 35 \cdot 6 + 21 = 231$

Method: Exactly 4 women or exactly 5 women — two disjoint cases, then add.

> **Investigate further:** A classic overcounting trap lurks here: "choose 4 women to guarantee the quota ($\binom{7}{4}$), then anyone from the remaining 9" gives $35 \cdot 9 = 315 \neq 231$. Find the committees counted twice by that argument — diagnosing it inoculates you permanently.

2. How many four-digit numbers have at least one repeated digit?

> **Answer:** $9000 - 9 \cdot 9 \cdot 8 \cdot 7 = 9000 - 4536 = 4464$

Method: Complement: all-distinct four-digit numbers number $9 \cdot 9 \cdot 8 \cdot 7$ (sheet 1 technique). Subtract from 9000.

> **Investigate further:** What fraction is that? Nearly half. Estimate the same fraction for 6-digit numbers, then investigate the birthday-paradox structure: with k digits drawn from 10, repeats dominate surprisingly early.

3. How many diagonals does a regular 12-gon have?

> **Answer:** $\binom{12}{2} - 12 = 66 - 12 = 54$

Method: Every pair of vertices gives a segment; remove the 12 sides. (Regularity is irrelevant — only convexity matters.)

> **Investigate further:** Derive the general formula $\frac{n(n-3)}{2}$ two ways: (i) $\binom{n}{2} - n$, (ii) each vertex sends $n-3$ diagonals and each is counted twice. Two derivations agreeing is your built-in error check.

4. Triangles from 8 points on a circle; how many avoid point A?

Answer: $\binom{8}{3} = 56$ in total; $\binom{7}{3} = 35$ avoid A

Method: All triples work (no three points on a circle are collinear). Avoiding A: all three vertices come from the other 7 points. Alternatively by complement: $56 - \binom{7}{2} = 56 - 21 = 35$ — subtract the triangles *through* A.

> **Investigate further:** Check by symmetry: each of the 56 triangles has 3 vertices, so vertex-uses total $56 \cdot 3 = 168 = 8 \cdot 21$ — each point is used 21 times, avoided 35 times, and $21 + 35 = 56$. Counting incidences two ways strikes again.

5. Teams of 6 from 6 boys and 6 girls with unequal numbers of each.

Answer: $\binom{12}{6} - \binom{6}{3}^2 = 924 - 400 = 524$

Method: Complement: the only *equal* split for a team of 6 is 3–3, counted by $\binom{6}{3}\binom{6}{3} = 400$. Subtract from all $\binom{12}{6} = 924$ teams. Listing the unequal splits (6–0, 5–1, 4–2 and mirrors) works too but is six cases instead of one subtraction.

> **Investigate further:** The identity behind the complement: $\sum_k \binom{6}{k}\binom{6}{6-k} = \binom{12}{6}$ — Vandermonde, proved by asking how many boys ended up in an unrestricted team. Verify the 6-case direct sum gives 524 and feel the difference in effort.

6. 5-card hands from 52 cards containing at least three aces.

Answer: $\binom{4}{3}\binom{48}{2} + \binom{48}{1} = 4512 + 48 = 4\,560$

Method: Two disjoint cases. *Exactly three aces:* choose which 3 of the 4 aces, then 2 non-aces: $4 \cdot 1128$. *All four aces:* the aces are forced, only the fifth card is free: 48.

> **Investigate further:** Why $\binom{4}{3}$ in one case but no $\binom{4}{4}$ anxiety in the other? Forced choices contribute a factor of 1. Extend: compute "at least two aces" and check all five ace-counts (0 through 4) sum to $\binom{52}{5}$.

7. A convex n-gon has exactly 27 diagonals. Find n.

Answer: $n = 9$

Method: $\frac{n(n-3)}{2} = 27 \Rightarrow n(n-3) = 54 = 9 \cdot 6 \Rightarrow n = 9$. (Product of two integers differing by 3 — read it off.)

> **Investigate further:** Which diagonal counts are achievable? Compute $\frac{n(n-3)}{2}$ for $n = 4, \ldots, 10$: $2, 5, 9, 14, 20, 27, 35$. Investigate: can two different polygons ever have the same diagonal count? (The sequence is strictly increasing — why does that settle it?)

8. Subsets of $\{1, \ldots, 10\}$ containing at least one of 1 and 2.

Answer: $2^{10} - 2^8 = 1024 - 256 = 768$

Method: Complement: subsets avoiding *both* 1 and 2 are exactly the subsets of the remaining 8 elements: 2^8.

> **Investigate further:** Recount by inclusion–exclusion: (contain 1) + (contain 2) − (contain both) $= 512 + 512 - 256 = 768$. You have met inclusion–exclusion informally — sheet 5 makes it a machine. Which route generalises better to "at least one of 1, 2, 3"?

Section D Challenge (SMC difficulty)

1. Rectangles on a 4×4 grid. *(extends Sheet 1, D2)*

Answer: $\binom{5}{2}^2 = 100$

Method: A rectangle is determined by choosing 2 of the 5 vertical gridlines and 2 of the 5 horizontal gridlines: $\binom{5}{2} \cdot \binom{5}{2} = 10 \cdot 10$. Sheet 1 counted the spans by listing $(4 + 3 + 2 + 1)$; the binomial coefficient *is* that sum.

> **Investigate further:** So $1 + 2 + 3 + 4 = \binom{5}{2}$ — triangular numbers are $\binom{n+1}{2}$ in disguise. Count the *squares* on the same grid $(16 + 9 + 4 + 1 = 30)$, then find how many rectangles on an 8×8 board are NOT squares.

2. Subsets of $\{1, \ldots, 8\}$ with no two consecutive integers (empty set included). *(cf. MAT 2023 Q5)*

Answer: 55

Method: Let a_n count such subsets of $\{1, \ldots, n\}$. Split on whether n is used: if not, a_{n-1} subsets; if yes, $n - 1$ is banned, leaving a_{n-2}. So $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 2$, $a_2 = 3$ — Fibonacci! The chain runs $2, 3, 5, 8, 13, 21, 34, 55$, so $a_8 = 55$.

> **Investigate further:** Alternative proof: a size-k subset with no neighbours corresponds to choosing k from $8 - k + 1$ positions after shrinking the gaps, giving $\sum_k \binom{9-k}{k} = 55$ — a Fibonacci number as a diagonal sum in Pascal's triangle. Both proofs preview sheet 7's recurrence toolkit.

3. Committees of any size from 6 juniors and 6 seniors with strictly more juniors.

Answer: $\dfrac{2^{12} - \binom{12}{6}}{2} = \dfrac{4096 - 924}{2} = 1\,586$

Method: Every committee has more juniors, more seniors, or a tie. Pair each junior with a senior once and for all; swapping every member for their partner is a bijection between "more juniors" committees and "more seniors" committees, so those two counts are equal. Ties (equal numbers, k of each) number $\sum_k \binom{6}{k}^2 = \binom{12}{6} = 924$ by Vandermonde. Hence each majority count is $\frac{4096-924}{2} = 1586$. No case-by-case summation needed.

> **Investigate further:** Case-sum check: $\sum_{j>s} \binom{6}{j}\binom{6}{s}$ really is 1586 — but the bijection argument scales to 100 juniors and 100 seniors untouched, while the sum does not. Warning: with *unequal* club sizes (7 juniors, 5 seniors) the swap bijection dies. What replaces it? (Nothing clean — that asymmetric version is a genuine slog, which is exactly why the symmetric version is the beautiful one.)

4. Six people each shake hands with exactly two others. In how many ways?

Answer: 70

Method: Everyone having exactly two handshakes forces the handshake pattern to be a union of cycles of length ≥ 3 covering all 6 people: either one 6-cycle or two 3-cycles. *One 6-cycle:* arrange 6 people in a cycle: $\frac{5!}{2} = 60$ (rotations and reflection give the same handshake set). *Two 3-cycles:* split into two trios: $\frac{1}{2}\binom{6}{3} = 10$, each trio shaking hands in exactly one way. Total $60 + 10 = 70$.

> **Investigate further:** Why is a "cycle" forced? Prove: a graph where every vertex has degree 2 is a disjoint union of cycles. Then redo the count for 7 people (answer: $\frac{6!}{2} + \binom{7}{3} \cdot 1 \cdot \frac{3!}{2} = 360 + 35 \cdot 3 = 465$ — careful: a 3-cycle plus a 4-cycle. Work it out and brute-check it).

5. Ten points on a circle, all chords drawn, no three concurrent inside. Interior intersection points?

Answer: $\binom{10}{4} = 210$

Method: The key insight: every interior intersection comes from exactly one set of 4 points on the circle (the two crossing chords are the diagonals of the quadrilateral those 4 points form), and every 4 points create exactly one interior crossing. So intersections $\leftrightarrow$ 4-subsets: $\binom{10}{4} = 210$. No coordinate geometry, no counting chords.

> **Investigate further:** This is a *bijection* doing all the work — sheet 7's crown tool. Follow-up: how many *triangles* are formed whose vertices are all interior intersection points... is much harder; instead count the regions the chords cut the disc into for $n = 1, \ldots, 6$ points $(1, 2, 4, 8, 16, 31!)$ and investigate why the doubling pattern breaks.

Top 5 Patterns Today

1. **Choose vs arrange.** $\binom{n}{r} = \frac{^nP_r}{r!}$: divide by $r!$ exactly when the selection is unordered. Ask "does swapping two chosen objects give a new outcome?"

2. **Symmetry flip.** $\binom{n}{r} = \binom{n}{n-r}$: always compute the smaller index. $\binom{100}{98}$ is a two-second question.

3. **Complementary counting.** "At least one X" = total − "no X". The complement of a hard condition is usually one clean product.

4. **Committee–chair double counting.** $r\binom{n}{r} = n\binom{n-1}{r-1}$: proving identities by counting a scenario two ways beats algebra.

5. **Objects $\leftrightarrow$ subsets.** Binary strings, lattice paths, handshakes, grid rectangles, chord crossings — all are "choose a subset" in costume. Find the subset.

Common Traps to Avoid

1. **Ordering an unordered selection.** A committee counted with ^nP_r is overcounted by a factor of $r!$. Decide ordered/unordered *before* touching numbers.

2. **The "guarantee the quota" overcount.** "Choose 4 women, then anyone else" double-counts committees with 5 women (C1). Quota problems need exact cases or a complement — never "reserve then top up".

3. **Adding overlapping cases.** Cases must be *disjoint* before you add. "Contains 1" and "contains 2" overlap; either subtract the overlap or complement.

4. **Forgetting forced choices are free.** All four aces in the hand is a factor of 1, not $\binom{4}{4}$ worth of anxiety — but *exactly* three aces needs $\binom{4}{3}$.

5. **Dividing by symmetry that isn't there.** Handshake circles divide by rotations *and* reflections ($\frac{5!}{2}$); labelled pair-splits divide by 2 only when the two groups are indistinguishable. Name the symmetry before dividing by it.

"When a count splits into cases, count the complement first — the opposite of 'at least one' is always simpler."

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths