Daily Algebra Drill #5

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Partial fractions with quadratic denominators · Elementary functional equations · Cauchy–Schwarz (Engel form) · Conjugate surd pairs revisited.

Attempt each question before checking answers. Time yourself. No calculators.

Section A Rapid Recognition (≤15 s each)

No expansion. Recognise the pattern, write the answer.

1. Simplify: $\dfrac{x^2-9}{x^2+5x+6}$.

2. Rationalise: $\dfrac{4}{\sqrt{7}-\sqrt{3}}$.

3. Given $x+y=3$ and $xy=-4$, find $(x-y)^2$.

4. Factorise: x^3-x^2-4x+4.

5. Without expanding, find the coefficient of x^2 in $(1+x)^4$.

6. Given $\alpha+\beta=-3$ and $\alpha\beta=-10$, write down the quadratic with roots α and β.

7. Simplify: $\sqrt{50}-\sqrt{18}+\sqrt{8}$.

8. Given $a+b=7$ and $a^2+b^2=29$, find a^3+b^3.

9. Solve: $|2x-3|=7$.

10. For $x>0$, find the minimum of $\dfrac{(x+4)^2}{x}$.

Section B Manipulation Drills (≤50 s each)

Fractions, structure, identities. Minimum working.

1. Decompose into partial fractions: $\dfrac{5x+1}{(x+1)(x-2)}$.

2. Decompose into partial fractions: $\dfrac{3}{x(x^2+3)}$.

3. Simplify: $\dfrac{1}{\sqrt{n}+\sqrt{n+1}}+\dfrac{1}{\sqrt{n+1}+\sqrt{n+2}}$ and hence evaluate $\displaystyle\sum_{k=0}^{8}\frac{1}{\sqrt{k}+\sqrt{k+1}}$.

4. Factorise: a^4+a^2+1 over $\mathbb{Z}$. (*Hint: add and subtract* a^2.)

5. Given $a+b=s$ and $ab=p$, express $(a-b)^4$ in terms of s and p.

6. Simplify: $\dfrac{x^2-2x+1}{x^2+2x-3}\div\dfrac{x^2-1}{x^2+4x+3}$.

7. The roots of $x^2+px+q=0$ are r and s. Find in terms of p and q the quadratic with roots r^2 and s^2.

8. Solve the simultaneous equations: $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{6}$ and $\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{6}$.

9. Prove: $\dfrac{1}{1\cdot3}+\dfrac{1}{3\cdot5}+\dfrac{1}{5\cdot7}+\cdots+\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{2n+1}$.

10. If $f(x)=\dfrac{ax+b}{cx+d}$ and $f(f(x))=x$ for all x, what condition must a,b,c,d satisfy?

Section C Substitution & Structure (≤90 s each)

One clean substitution is worth ten lines of algebra.

1. Solve: $\left(x^2+\dfrac{1}{x^2}\right)-7\left(x-\dfrac{1}{x}\right)-10=0$.

2. For positive reals a,b,c with $a+b+c=3$, prove $\dfrac{a^2}{b}+\dfrac{b^2}{c}+\dfrac{c^2}{a}\geq 3$ using the Cauchy–Schwarz (Engel/Titu) form: $\dfrac{x^2}{p}+\dfrac{y^2}{q}+\dfrac{z^2}{r}\geq\dfrac{(x+y+z)^2}{p+q+r}$.

3. Solve the system: $x^2+xy=12,\quad xy+y^2=6$.

4. Find the range of values of k for which $kx^2-4x+k-3=0$ has two distinct real roots.

5. Simplify $\sqrt{3+2\sqrt{2}}+\sqrt{3-2\sqrt{2}}$ without a calculator.

6. If $x-\dfrac{1}{x}=4$, find the exact value of $\sqrt{x}-\dfrac{1}{\sqrt{x}}$.

7. Given $a,b>0$ with $\dfrac{1}{a}+\dfrac{1}{b}=1$, show that $(a-1)(b-1)=1$ and find the minimum of $a+b$.

8. Solve: $(x^2-x-1)^2=(2x+1)^2$.

Section D Challenge (SMC / BMO1 difficulty)

No brute force. Each question rewards one key observation.

1. A function f satisfies $f(x) + f\left(\dfrac{1}{1-x}\right) = x$ for all $x \neq 0, 1$. Find $f(x)$. (*Apply the substitution twice more to get three equations.*)

2. For positive reals, prove the Cauchy–Schwarz inequality in **Engel (Titu) form**:

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}.$$

Hence find the minimum of $\dfrac{1}{x} + \dfrac{4}{1-x}$ for $0 < x < 1$.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x+y) = f(x) + f(y)$ and $f(1) = 3$. (*First find f on $\mathbb{Z}$, then state any assumptions needed for $\mathbb{R}$.*)

4. Show that among any 5 integers, there exist at least 2 whose difference is divisible by 4. Hence prove that for any 5 integers, two of them satisfy $a^2 \equiv b^2 \pmod 4$.

5. Solve for integers: $x^2 - y^2 = 2024$.

"A formula you have derived yourself is worth a hundred you have memorised."

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths