Daily Algebra Drill #3

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Remainder & Factor Theorem · AM–GM & algebraic inequalities · Completing the square in two variables · Rational Root Theorem · Modular/divisibility arguments · Parametric families of solutions.

Attempt each question before checking answers. Time yourself. No calculators.

Section A Rapid Recognition (≤15 s each)

Zero expansion. Pure pattern recognition.

1. The polynomial $f(x) = x^3 - 3x^2 + kx - 8$ has $(x-2)$ as a factor. Find k.

2. Without dividing, find the remainder when $p(x) = x^{100} - 1$ is divided by $(x-1)$.

3. Factorise: $6x^2 - x - 12$.

4. Simplify: $\dfrac{x^{2n}-1}{x^n-1}$ $\quad (n \in \mathbb{Z}^+)$.

5. Given $x > 0$, find the minimum value of $x + \dfrac{9}{x}$.

6. Factorise over $\mathbb{Z}$: $x^4 - y^4$.

7. The roots of $2x^2 - 6x + 3 = 0$ are α and β. Find $\alpha^2 + \beta^2$.

8. Simplify: $\dfrac{(n+1)! - n!}{(n-1)!}$ $\quad (n \geq 1)$.

9. Without expanding, show that $(a+b+c)^2 - (a^2+b^2+c^2) = 2(ab+bc+ca)$ and state the value of $ab+bc+ca$ when $a+b+c = 4$ and $a^2+b^2+c^2 = 8$.

10. Find the value of k such that $x^2 + kx + 16$ is a perfect square.

Section B Manipulation Drills (≤50 s each)

Fractions, rearranging, hidden factorisations. Minimal expansion.

1. Find all rational roots of $2x^3 - 3x^2 - 11x + 6$, and hence factorise it completely.

2. Simplify: $\dfrac{x^3+1}{x^2-x+1}$.

3. Given $a+b=p$ and $a^3+b^3=q$, express ab in terms of p and q.

4. For $x \neq 0$, simplify: $\dfrac{x^2(x+1)-x(x+1)^2}{x(x+1)}$.

5. Factorise: $a^3-a^2b-ab^2+b^3$.

6. The sum $\displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$. Use this to find, without expanding, $\displaystyle\sum_{r=1}^{n}(2r-1)$.

7. Solve for x: $\dfrac{2x-1}{x+3}-\dfrac{x+2}{x-1}=1$.

8. Simplify: $\left(\dfrac{1}{a}-\dfrac{1}{b}\right) \div \left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)$.

9. Show that n^3-n is divisible by 6 for all integers n.

10. Given α, β are roots of $x^2-x-1=0$, find $\alpha^3+\beta^3$ and $\alpha^4+\beta^4$ without solving for α or β.

Section C Substitution & Structure ($\leq$90 s each)

Find structure before computing. One good substitution beats ten lines of algebra.

1. Solve: $9^x-10\cdot 3^x+9=0$.

2. Find the minimum value of x^2+y^2 subject to $x+2y=5$.

3. Prove that $a^2+b^2+c^2 \geq ab+bc+ca$ for all real a, b, c.

4. Solve: $\sqrt{x+5}+\sqrt{x-3}=4$.

5. The expression $x^2+4x+y^2-6y+13$ has a minimum value. Find it, and the values of x and y at which it is achieved.

6. If $x+\dfrac{1}{x}=5$, find $\dfrac{x^4+1}{x^2}$.

7. Solve over the reals: $x^4+4x^2-5=0$.

8. Given $a, b>0$ with $a+b=1$, find the minimum of $\dfrac{1}{a}+\dfrac{1}{b}$.

Section D Challenge (TMUA / SMC difficulty)

Five lines or fewer. Find the structure, not the computation.

1. Find all real solutions to $x^4+2x^3-2x-1=0$. (*Hint: the left-hand side factorises into two quadratics.*)

2. Prove that for all integers n, the expression $n^2 + n + 41$ is not always prime by finding the smallest counterexample. Then explain why $n^2 + n + 41$ is prime for $n = 0, 1, 2, \ldots, 39$.

3. Let $a, b, c > 0$ with $abc = 1$. Prove that $a + b + c \geq 3$.

4. For positive reals a and b, prove that

$$\frac{a}{b} + \frac{b}{a} \geq 2.$$

Hence prove the AM–GM inequality $\dfrac{a+b}{2} \geq \sqrt{ab}$ for $a, b > 0$.

5. Find the number of integers n with $1 \leq n \leq 1000$ such that $n^2 - 1$ is divisible by $n + 3$.

"Mathematics is not about numbers, equations or algorithms: it is about understanding."
— William Paul Thurston

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths