Daily Algebra Drill #5 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Partial fractions with quadratic denominators · Elementary functional equations · Cauchy–Schwarz (Engel form) · Conjugate surd pairs revisited.

Section A Rapid Recognition

1. Simplify $\dfrac{x^2-9}{x^2+5x+6}$.

Answer: $\dfrac{x-3}{x+2}$

Method: $(x-3)(x+3)/[(x+2)(x+3)]$. Cancel $(x+3)$. Undefined at $x=-3$ and $x=-2$.

> **Investigate further:** The cancelled factor $(x+3)$ creates a *removable discontinuity* (hole) at $x=-3$. Sketch the graph of both the original and simplified form. What is the y-value of the hole?

2. Rationalise: $\dfrac{4}{\sqrt{7}-\sqrt{3}}$.

Answer: $\sqrt{7}+\sqrt{3}$

Method: Multiply by $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}+\sqrt{3}}$: denominator $=7-3=4$. Cancel the 4.

> **Investigate further:** Rationalising is conjugate multiplication. The pattern $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})=a-b$ applies whenever you see a two-term surd denominator. Use it to simplify $\dfrac{1}{\sqrt{n+1}-\sqrt{n}}$ and telescope $\sum_{n=1}^{99}\dfrac{1}{\sqrt{n+1}-\sqrt{n}}$.

3. Given $x+y=3$, $xy=-4$, find $(x-y)^2$.

Answer: 25

Method: $(x-y)^2=(x+y)^2-4xy=9+16=25$.

> **Investigate further:** So $x-y=\pm 5$. Combined with $x+y=3$: $x=4, y=-1$ or $x=-1, y=4$. Verify $xy=-4$. ✓Now: without finding x and y, find $x^3-y^3=(x-y)(x^2+xy+y^2)$.

4. Factorise: x^3-x^2-4x+4.

Answer: $(x-1)(x-2)(x+2)$

Method: Group: $x^2(x-1)-4(x-1)=(x-1)(x^2-4)=(x-1)(x-2)(x+2)$.

> **Investigate further:** The grouping $x^2(x-1) - 4(x-1)$ works because both groups share the factor $(x-1)$. For which cubics $x^3 + ax^2 + bx + c$ does this grouping strategy work? (When $c = ab$, so that $x^3 + ax^2 + bx + ab = x^2(x+a) + b(x+a)$.) Verify for this example: $a = -1$, $b = -4$, $c = 4 = (-1)(-4)$. ✓

5. Without expanding, find the coefficient of x^2 in $(1 + x)^4$.

Answer: 6

Method: $\binom{4}{2} = 6$.

> **Investigate further:** Pascal's triangle row 4: $1, 4, 6, 4, 1$. Memorise rows 0–6. Then: find the coefficient of x^3 in $(2 + 3x)^5$ using the binomial theorem: $\binom{5}{3}(2)^2(3x)^3 = 10 \cdot 4 \cdot 27x^3 = 1080x^3$.

6. Write the quadratic with roots α, β where $\alpha + \beta = -3$, $\alpha\beta = -10$.

Answer: $x^2 + 3x - 10 = 0$

Method: $x^2 - (\alpha + \beta)x + \alpha\beta = x^2 - (-3)x + (-10) = x^2 + 3x - 10$.

> **Investigate further:** Factorise: $(x + 5)(x - 2) = 0$, roots -5 and 2. Check sum $= -3$, product $= -10$. ✓Now find the quadratic with roots α^2 and β^2 using only the sums and products (without finding α, β): sum $= \alpha^2 + \beta^2 = 9 + 20 = 29$, product $= 100$.

7. Simplify: $\sqrt{50} - \sqrt{18} + \sqrt{8}$.

Answer: $4\sqrt{2}$

Method: $5\sqrt{2} - 3\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$.

> **Investigate further:** Always reduce surds to $k\sqrt{n}$ with n square-free before combining. The technique: $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$. Simplify $\sqrt{75} + \sqrt{48} - \sqrt{27}$ and $\sqrt{200} - \sqrt{162} + \sqrt{32}$.

8. Given $a + b = 7$, $a^2 + b^2 = 29$, find $a^3 + b^3$.

Answer: 133

Method: $ab = \frac{(a+b)^2 - (a^2+b^2)}{2} = \frac{49-29}{2} = 10$. Then $a^3 + b^3 = 7(29 - 10) = 7 \times 19 = 133$.

> **Investigate further:** Note $133 = 7 \times 19$. Also $a^3 + b^3 = (a + b)[(a + b)^2 - 3ab] = 7(49 - 30) = 7 \times 19$. Verify $a = 5, b = 2$ satisfies both conditions and $5^3 + 2^3 = 133$. ✓

9. Solve: $|2x - 3| = 7$.

Answer: $x = 5$ or $x = -2$

Method: $2x - 3 = 7 \Rightarrow x = 5$; $2x - 3 = -7 \Rightarrow x = -2$. Draw the V-shape: $y = |2x - 3|$ meets $y = 7$ at two points.

> **Investigate further:** The solution set $\{x : |2x - 3| = 7\} = \{-2, 5\}$. Now solve $|2x - 3| = |x + 4|$ by splitting into cases $2x - 3 = x + 4$ and $2x - 3 = -(x + 4)$.

10. Find the minimum of $\dfrac{(x+4)^2}{x}$ for $x > 0$.

Answer: 16

Method: $\frac{(x+4)^2}{x} = x + 8 + \frac{16}{x} \geq 2\sqrt{16} + 8 = 16$ by AM–GM on $x + \frac{16}{x} \geq 8$. Equality at $x = 4$.

> **Investigate further:** Alternatively write $\frac{(x+4)^2}{x} = (\sqrt{x} + \frac{4}{\sqrt{x}})^2$. Then minimise $\sqrt{x} + \frac{4}{\sqrt{x}}$ using AM–GM: $\geq 2\sqrt{4} = 4$. Squaring: ≥ 16. This substitution approach ($t = \sqrt{x}$) is faster for recognising the perfect square form.

Section B Manipulation Drills

1. Partial fractions: $\dfrac{5x+1}{(x+1)(x-2)}$.

Answer: $\dfrac{4/3}{x+1} + \dfrac{11/3}{x-2}$, or $\dfrac{4}{3(x+1)} + \dfrac{11}{3(x-2)}$

Method: $5x + 1 = A(x-2) + B(x+1)$. $x = 2$: $11 = 3B \Rightarrow B = \frac{11}{3}$. $x = -1$: $-4 = -3A \Rightarrow A = \frac{4}{3}$.

> **Investigate further:** Partial fractions are the algebraic inverse of combining fractions. They are essential for integrating rational functions and for telescoping sums. Now decompose $\dfrac{2x^2+3}{(x-1)(x+1)(x+2)}$ — note the degree of numerator is less than the degree of denominator, so the form is $\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$.

2. Partial fractions: $\dfrac{3}{x(x^2+3)}$.

Answer: $\dfrac{1}{x} - \dfrac{x}{x^2+3}$

Method: $3 = A(x^2+3) + (Bx+C)x$. $x = 0$: $3 = 3A \Rightarrow A = 1$. Coefficient of x^2: $0 = A + B \Rightarrow B = -1$. Coefficient of x: $0 = C$. Result: $\frac{1}{x} - \frac{x}{x^2+3}$.

> **Investigate further:** The irreducible quadratic denominator $x^2 + 3$ has roots $\pm i\sqrt{3}$, which are complex. This means we *cannot* factor further over $\mathbb{R}$, so the numerator over $x^2 + 3$ must be linear ($Bx + C$). This is the key rule: linear numerator over irreducible quadratic. Practice with $\dfrac{x^2+1}{x(x^2+x+1)}$.

3. Show $\dfrac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$; hence evaluate $\sum_{k=0}^{8} \dfrac{1}{\sqrt{k}+\sqrt{k+1}}$.

Answer: 3

Method: Rationalise: $\frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n} = \sqrt{n+1} - \sqrt{n}$. The sum telescopes: $(\sqrt{1} - \sqrt{0}) + (\sqrt{2} - \sqrt{1}) + \cdots + (\sqrt{9} - \sqrt{8}) = \sqrt{9} - \sqrt{0} = 3$.

> **Investigate further:** This is the classic rationalise-then-telescope technique. Now evaluate $\sum_{k=1}^{n} \dfrac{1}{\sqrt{k}+\sqrt{k+1}}$ in closed form (answer: $\sqrt{n+1} - 1$), and find the smallest n for which this sum exceeds 9.

4. Factorise $a^4 + a^2 + 1$ over $\mathbb{Z}$.

Answer: $(a^2 + a + 1)(a^2 - a + 1)$

Method: Add and subtract a^2: $a^4 + 2a^2 + 1 - a^2 = (a^2+1)^2 - a^2 = (a^2+a+1)(a^2-a+1)$.

> **Investigate further:** Note $a^4 + a^2 + 1 = \frac{a^6-1}{a^2-1}$ (geometric series), so this factorisation reflects the cyclotomic structure of $x^6 - 1$. Verify that the roots of $a^2 + a + 1 = 0$ and $a^2 - a + 1 = 0$ are the primitive 3rd and 6th roots of unity respectively.

5. Express $(a-b)^4$ in terms of $s = a+b$ and $p = ab$.

Answer: $(s^2 - 4p)^2$

Method: $(a-b)^2 = (a+b)^2 - 4ab = s^2 - 4p$. Then $(a-b)^4 = [(a-b)^2]^2 = (s^2-4p)^2$.

> **Investigate further:** Find $(a-b)^6$ and $(a-b)^8$ in terms of s and p. Note the pattern: $(a-b)^{2k} = (s^2-4p)^k$. Now express $a^4 + b^4$ in terms of s and p: $a^4 + b^4 = (a^2+b^2)^2 - 2a^2b^2 = [(s^2-2p)^2 - 2p^2]$.

6. Simplify $\dfrac{x^2-2x+1}{x^2+2x-3} \div \dfrac{x^2-1}{x^2+4x+3}$.

Answer: 1

Method: Factor everything: $\frac{(x-1)^2}{(x-1)(x+3)} \cdot \frac{(x+1)(x+3)}{(x-1)(x+1)} = \frac{x-1}{x+3} \cdot \frac{x+3}{x-1} = 1$.

> **Investigate further:** This simplification to 1 means the two rational expressions are reciprocals of each other. Verify by noting that the second fraction is the reciprocal of the first (after simplification). For what values of x is the original expression undefined?

7. Find the quadratic with roots r^2 and s^2 where r, s are roots of $x^2+px+q=0$.

Answer: $x^2 - (p^2-2q)x + q^2 = 0$

Method: $r^2 + s^2 = (r+s)^2 - 2rs = p^2 - 2q$. $(rs)^2 = q^2$. Quadratic: $x^2 - (p^2-2q)x + q^2 = 0$.

> **Investigate further:** This is the Tschirnhaus substitution idea: transforming the roots without solving for them. Now find the quadratic whose roots are r^3, s^3 when r, s satisfy $x^2 - 3x + 1 = 0$. (Hint: use the fact that $r^3 + s^3$ and r^3s^3 can be found from the Vieta data.)

8. Solve: $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6}$ and $\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{6}$.

Answer: $x = 2,\ y = 3$

Method: Let $u = \frac{1}{x}$, $v = \frac{1}{y}$: $u + v = \frac{5}{6}$, $u - v = \frac{1}{6}$. Adding: $2u = 1 \Rightarrow u = \frac{1}{2}$, $v = \frac{1}{3}$.

> **Investigate further:** This substitution ($u = 1/x$) converts a non-linear system to a linear one. The same trick works for systems involving $\frac{1}{x^2}$ (substitute $u = \frac{1}{x^2}$). Now solve: $\frac{2}{x} + \frac{3}{y} = 7$ and $\frac{4}{x} - \frac{1}{y} = 1$.

9. Prove: $\dfrac{1}{(2k-1)(2k+1)} = \dfrac{1}{2}\left(\dfrac{1}{2k-1} - \dfrac{1}{2k+1}\right)$; hence $\sum_{k=1}^{n} \dfrac{1}{(2k-1)(2k+1)} = \dfrac{n}{2n+1}$.

Answer: Proof by partial fractions, then telescoping.

Method: PF: $\frac{1}{(2k-1)(2k+1)} = \frac{A}{2k-1} + \frac{B}{2k+1}$. Solving: $A = \frac{1}{2}$, $B = -\frac{1}{2}$. Sum telescopes: $\frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}$. $\square$

> **Investigate further:** The denominator $(2k-1)(2k+1)$ is the product of two consecutive odd numbers. The partial fraction structure is $\frac{1}{2d}\left(\frac{1}{a}-\frac{1}{b}\right)$ where $b-a=d$. This generalises: $\frac{1}{(n+a)(n+b)} = \frac{1}{b-a}\left(\frac{1}{n+a}-\frac{1}{n+b}\right)$. Use this to find $\sum_{k=1}^{n} \frac{1}{(k+1)(k+3)}$.

10. If $f(x) = \dfrac{ax+b}{cx+d}$ and $f(f(x)) = x$, what condition must a, b, c, d satisfy?

Answer: $a+d=0$ (equivalently $d=-a$)

Method: Compute $f(f(x)) = \frac{a \cdot f(x)+b}{c \cdot f(x)+d} = \frac{(a^2+bc)x+(a+d)b}{(a+d)cx+(bc+d^2)}$. For this to equal x: need $a^2+bc = bc+d^2$ (i.e. $a^2=d^2$) and $(a+d)b=0$ and $(a+d)c=0$. If $a=-d$, all conditions hold automatically.

> **Investigate further:** Such a function is called an *involution* or a *Möbius involution*. Examples: $f(x) = \frac{1}{x}$ ($a=0, d=0$ degenerate), $f(x) = \frac{-x+1}{x+1}$ (check $a+d=0$). Find two more Möbius involutions and verify $f(f(x)) = x$ for each.

Section C Substitution & Structure

1. Solve: $\left(x^2+\dfrac{1}{x^2}\right) - 7\left(x-\dfrac{1}{x}\right) - 10 = 0.$

Answer: $x - \dfrac{1}{x} = 8$ (giving $x = 4 \pm \sqrt{17}$) or $x - \dfrac{1}{x} = -1$ (giving $x = \frac{-1\pm\sqrt{5}}{2}$)

Method: Let $u = x - \frac{1}{x}$, so $x^2 + \frac{1}{x^2} = u^2+2$. Then $(u^2+2) - 7u - 10 = 0 \Rightarrow u^2 - 7u - 8 = 0 \Rightarrow (u-8)(u+1) = 0$. $u=8$: $x^2-8x-1=0 \Rightarrow x = 4\pm\sqrt{17}$. $u=-1$: $x^2+x-1=0 \Rightarrow x = \frac{-1\pm\sqrt{5}}{2}$.

> **Investigate further:** The lesson: when you see $x^2+\frac{1}{x^2}$ *alongside* $x-\frac{1}{x}$, use $u = x-\frac{1}{x}$ (since $u^2 = x^2-2+\frac{1}{x^2}$). When you see $x^2+\frac{1}{x^2}$ *alongside* $x+\frac{1}{x}$, use $v = x+\frac{1}{x}$ instead. Carefully distinguish which combination is present before substituting.

2. Using the Engel (Titu) form of Cauchy–Schwarz, prove $\dfrac{a^2}{b}+\dfrac{b^2}{c}+\dfrac{c^2}{a} \geq 3$ when $a+b+c = 3$.

Answer: Proved below.

Method: Titu: $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq \frac{(a+b+c)^2}{b+c+a} = \frac{9}{3} = 3$. $\square$ Equality iff $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$, i.e. $a=b=c=1$.

> **Investigate further:** The Engel/Titu form states $\sum \frac{x_i^2}{y_i} \geq \frac{(\sum x_i)^2}{\sum y_i}$. Prove it from the standard Cauchy–Schwarz $(\sum u_i v_i)^2 \leq (\sum u_i^2)(\sum v_i^2)$ by setting $u_i = \frac{x_i}{\sqrt{y_i}}$ and $v_i = \sqrt{y_i}$. This form is indispensable in olympiad inequality problems.

3. Solve the system: $x^2+xy = 12,\ \ xy+y^2 = 6$.

Answer: $(x,y) = (2\sqrt{2},\ \sqrt{2})$ and $(-2\sqrt{2},\ -\sqrt{2})$

Method: Factor both left sides: $x(x+y) = 12$ and $y(x+y) = 6$. Neither y nor $x+y$ can be zero (each would force $0=6$ or $0=12$), so divide: $\frac{x}{y} = 2$, giving $x = 2y$. Substitute into $xy+y^2 = 6$: $2y^2+y^2 = 6 \Rightarrow y^2 = 2 \Rightarrow y = \pm\sqrt{2}$, with $x = 2y$ matching the sign. Check: $8+4 = 12$ ✓, $4+2 = 6$ ✓.

> **Investigate further:** Dividing equations is a powerful technique when the ratio reveals a simple relationship. But division requires care: we must check we're not dividing by zero. When might

$xy + y^2 = 0$? Then $y = 0$ or $y = -x$; check these cases separately.

4. Find values of k for which $kx^2 - 4x + k - 3 = 0$ has two distinct real roots.

Answer: $-1 < k < 4$ and $k \neq 0$

Method: For a quadratic ($k \neq 0$): discriminant $= 16 - 4k(k-3) = 16 - 4k^2 + 12k > 0 \Rightarrow 4k^2 - 12k - 16 < 0 \Rightarrow k^2 - 3k - 4 < 0 \Rightarrow (k-4)(k+1) < 0$. So $-1 < k < 4$. (Exclude $k = 0$ since that makes it linear, not quadratic.)

Investigate further: The discriminant condition is the standard approach, but always check the leading coefficient separately. Now find the values of k for which $kx^2 - 4x + k - 3 = 0$ has *equal* roots (one repeated real root). (Answer: $k = -1$ or $k = 4$.)

5. Simplify $\sqrt{3 + 2\sqrt{2}} + \sqrt{3 - 2\sqrt{2}}$.

Answer: $2\sqrt{2}$

Method: $3 + 2\sqrt{2} = (\sqrt{2} + 1)^2$, so $\sqrt{3 + 2\sqrt{2}} = \sqrt{2} + 1$. Similarly $3 - 2\sqrt{2} = (\sqrt{2} - 1)^2 \Rightarrow \sqrt{3 - 2\sqrt{2}} = \sqrt{2} - 1$. Sum $= 2\sqrt{2}$.

Investigate further: The denesting technique: to simplify $\sqrt{a + b\sqrt{c}}$, try $\sqrt{p} + \sqrt{q}$ where $p + q = a$ and $4pq = b^2c$. For $\sqrt{3 - 2\sqrt{2}}$: $p + q = 3$, $4pq = 8 \Rightarrow pq = 2$. So p, q are roots of $t^2 - 3t + 2 = (t-1)(t-2) = 0$: $p = 2, q = 1$. Hence $\sqrt{2} - 1$ (taking positive root). Apply this to $\sqrt{7 + 4\sqrt{3}}$.

6. If $x - \dfrac{1}{x} = 4$, find $\sqrt{x} - \dfrac{1}{\sqrt{x}}$ exactly.

Answer: $\sqrt{x} - \dfrac{1}{\sqrt{x}} = \sqrt{2\sqrt{5} - 2}$.

Method: Square the given relation: $(x - \frac{1}{x})^2 = 16 \Rightarrow x + \frac{1}{x} = 2\sqrt{5}$ (positive since $x > 0$). Let $t = \sqrt{x} - \frac{1}{\sqrt{x}}$. Then $t^2 = x + \frac{1}{x} - 2 = 2\sqrt{5} - 2$, so $t = \sqrt{2\sqrt{5} - 2}$ (positive branch).

Investigate further: The substitution $t = \sqrt{x} - 1/\sqrt{x}$ is a "half-power" reduction. Note the pattern: $x - 1/x = t(\underbrace{\sqrt{x} + 1/\sqrt{x}}_{\sqrt{t^2+4}})$. This generalises: to go from $x \pm 1/x$ to $x^{1/2} \pm 1/x^{1/2}$, use this factorisation chain. Apply it to find $\sqrt[4]{x} - 1/\sqrt[4]{x}$ when $x - 1/x = \sqrt{5}$.

7. Given $a, b > 0$ with $\frac{1}{a} + \frac{1}{b} = 1$, show $(a-1)(b-1) = 1$ and find $\min(a + b)$.

Answer: $(a-1)(b-1) = 1$ always; minimum of $a + b$ is 4, at $a = b = 2$.

Method: From $\frac{1}{a} + \frac{1}{b} = 1$: $a + b = ab$. $(a-1)(b-1) = ab - a - b + 1 = (a+b) - (a+b) + 1 = 1$. $\square$ For minimum of $a + b = ab$: AM–GM gives $ab \leq \left(\frac{a+b}{2}\right)^2$, so $a + b \leq \frac{(a+b)^2}{4}$, giving $a + b \geq 4$, with equality at $a = b = 2$.

Investigate further: The constraint $\frac{1}{a} + \frac{1}{b} = 1$ defines a hyperbola in the (a, b)-plane restricted to $a, b > 0$. The minimum of $a + b$ subject to this constraint (answer: 4 at $a = b = 2$) can also be found by substitution $b = \frac{a}{a-1}$ and then minimising $a + \frac{a}{a-1}$. Try this approach.

8. Solve: $(x^2 - x - 1)^2 = (2x + 1)^2$.

Answer: $x = 0,\ x = -1,\ x = \dfrac{3 \pm \sqrt{17}}{2}$

Method: $A^2 = B^2 \Leftrightarrow (A - B)(A + B) = 0$. $A - B = x^2 - 3x - 2$ and $A + B = x^2 + x = x(x + 1)$. $x(x + 1) = 0$: $x = 0$ or $x = -1$. $x^2 - 3x - 2 = 0$: $x = \frac{3\pm\sqrt{17}}{2}$. Four real solutions in total.

> **Investigate further:** The key move: $A^2 = B^2 \Leftrightarrow (A-B)(A+B) = 0$. Never expand both squares and combine — always factor as a difference of two squares first. Use this to solve $|(x^2-2)^2-(x-1)^2| = 0$ and $(x^2 + x)^2 = (x - 1)^2$.

Section D Challenge

(SMC / BMO1 difficulty)

1. Find $f(x)$ given $f(x) + f\left(\frac{1}{1-x}\right) = x$.

Answer: $f(x) = \dfrac{x^3 - x + 1}{2x(x - 1)}$

Method: Let the substitution $\phi(x) = \frac{1}{1-x}$. Note $\phi(\phi(x)) = \phi\left(\frac{1}{1-x}\right) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{-x} = \frac{x-1}{x}$ and $\phi^3(x) = x$ (three-cycle). So the three equations are: $f(x) + f\left(\frac{1}{1-x}\right) = x$ (i) $f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = \frac{1}{1-x}$ (ii) $f\left(\frac{x-1}{x}\right)+f(x) = \frac{x-1}{x}$ (iii) Add all three and divide by 2: $f(x)+f\left(\frac{1}{1-x}\right)+f\left(\frac{x-1}{x}\right) = \frac{1}{2}\left(x + \frac{1}{1-x} + \frac{x-1}{x}\right)$. Then subtract (ii): $f(x) = \frac{1}{2}\left(x + \frac{1}{1-x} + \frac{x-1}{x}\right) - \frac{1}{1-x} = \frac{1}{2}\left(x - \frac{1}{1-x} + \frac{x-1}{x}\right) = \frac{x^3-x+1}{2x(x-1)}$. Check at $x = 2$: $f(2) + f(-1) = \frac{7}{4} + \frac{1}{4} = 2$. ✓

> **Investigate further:** The three-cycle $x \to \frac{1}{1-x} \to \frac{x-1}{x} \to x$ is a well-known Möbius transformation cycle. Any functional equation on such a cycle can be solved by the add-then-subtract method. Find all f satisfying $f(x) + 2f(1 - x) = 3x$.

2. Prove the Cauchy–Schwarz inequality in Engel form; find $\min\left(\frac{1}{x} + \frac{4}{1-x}\right)$ for $0 < x < 1$.

Answer: Minimum is 9 at $x = \frac{1}{3}$.

Method: *Proof:* By standard C–S: $\left(\sum \frac{a_i^2}{b_i}\right)(\sum b_i) \geq (\sum a_i)^2$, giving $\sum \frac{a_i^2}{b_i} \geq \frac{(\sum a_i)^2}{\sum b_i}$. Apply with $a_1 = 1$, $b_1 = x$, $a_2 = 2$, $b_2 = 1 - x$: $\frac{1}{x} + \frac{4}{1-x} \geq \frac{(1+2)^2}{x+(1-x)} = 9$. Equality when $\frac{1/1}{x/(1)} = \frac{2/(1)}{(1-x)/1}$, i.e. $\frac{1}{x} = \frac{2}{1-x} \Rightarrow x = \frac{1}{3}$.

> **Investigate further:** The Engel form inequality $\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$ is equivalent to $(ay - bx)^2 \geq 0$. Prove this algebraically. Then use the Engel form to prove that for positive a, b, c: $\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}$ (Nesbitt variant).

3. Find all $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x + y) = f(x) + f(y)$ and $f(1) = 3$.

Answer: $f(x) = 3x$ for $x \in \mathbb{Q}$; with continuity assumption, $f(x) = 3x$ for all $x \in \mathbb{R}$.

Method: $f(n) = 3n$ for $n \in \mathbb{Z}$ (by induction). $f(p/q) = 3(p/q)$ for $p/q \in \mathbb{Q}$ (from $q \cdot f(p/q) = f(p) = 3p$). Without continuity or monotonicity, pathological solutions exist on $\mathbb{R}$ (via Hamel basis). With any regularity assumption: $f(x) = 3x$.

> **Investigate further:** Cauchy's functional equation $f(x+y) = f(x) + f(y)$ (Cauchy 1821) is the first example where "algebraic" solutions and "analytic" solutions differ. Explore: what happens if we additionally require f to be monotone? Or bounded on an interval? Both conditions force $f(x) = cx$.

4. Show that among any 5 integers there exist 2 whose difference is divisible by 4. Hence prove that for any 5 integers, two of them satisfy $a^2 \equiv b^2 \pmod 4$.

Answer: Proved below.

Method: Every integer has residue $0, 1, 2,$ or $3 \pmod 4$ (four classes). By the Pigeonhole Principle, among 5 integers, at least two share the same residue. Their difference $\equiv 0 \pmod 4$. $\square$ For the second part: $a \equiv b \pmod 4 \Rightarrow a^2 - b^2 = (a-b)(a+b)$. Since $4 \mid (a-b)$, we have $4 \mid (a^2 - b^2)$, i.e. $a^2 \equiv b^2 \pmod 4$.

> **Investigate further: Pigeonhole Principle:** if n objects are placed into $k < n$ boxes, at least one box contains at least $\lceil n/k \rceil$ objects. This is one of the most powerful tools in combinatorics and number theory. Use it to prove: among any 13 integers, two have the same remainder when divided by 12.

5. Find all integer solutions to $x^2 - y^2 = 2024$.

Answer: $(x, y) = (\pm 507, \pm 505)$, $(\pm 255, \pm 251)$, $(\pm 57, \pm 35)$, $(\pm 45, \pm 1)$ — all sign combinations, 16 solutions in total.

Method: $(x-y)(x+y) = 2024 = 2^3 \times 11 \times 23$. Write $x - y = a$, $x + y = b$ where $ab = 2024$ and a, b have the same parity (both even, since 2024 is even). Set $a = 2s$, $b = 2t$: $4st = 2024 \Rightarrow st = 506 = 2 \times 11 \times 23$. Factor pairs of 506 (both positive, $s \le t$): $(1, 506), (2, 253), (11, 46), (22, 23)$. Each gives $x = s + t$, $y = t - s$. Include negative values of x, y.

> **Investigate further:** $x^2 - y^2 = N$ has integer solutions iff N can be written as a product of two integers of the same parity. For odd N: every factorisation $N = ab$ works (since odd×odd = odd). For $N \equiv 2 \pmod 4$: no solutions (since $x^2 - y^2 = (x-y)(x+y)$ and $x \pm y$ share parity, so the product is $\equiv 0$ or $3 \pmod 4$, never 2). For $N \equiv 0 \pmod 4$: use even factorisations. Classify all N for which $x^2 - y^2 = N$ has no solutions.

Top 5 Patterns — Worksheet #5

1. **Partial fractions with irreducible quadratic denominator.** Use $\frac{Bx+C}{x^2+px+q}$ (linear numerator) when the denominator has negative discriminant.
2. **Telescoping via rationalisation.** $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$. Memorise this — sums telescope to $\sqrt{N} - \sqrt{0}$.
3. **Engel/Titu form of Cauchy–Schwarz.** $\sum \frac{a_i^2}{b_i} \ge \frac{(\sum a_i)^2}{\sum b_i}$. The fastest route to many optimisation inequalities.
4. **Möbius involutions: $d = -a$.** If $f(x) = \frac{ax+b}{cx+d}$ and $d = -a$, then $f(f(x)) = x$. Recognise this structure immediately.
5. **$A^2 = B^2 \Rightarrow (A-B)(A+B) = 0$.** Never expand both sides of a squared equation — always rewrite as a difference of two squares first.

Common Traps — Worksheet #5

1. **Using a linear numerator for reducible quadratic denominators.** $x^2 - 1 = (x-1)(x+1)$ is reducible — use $\frac{A}{x-1} + \frac{B}{x+1}$, not $\frac{Bx+C}{x^2-1}$.

2. **Forgetting the parity condition in $x^2 - y^2 = N$.** When $N \equiv 2 \pmod 4$, there are no integer solutions. Check this first.

3. **Mixing $x + 1/x$ and $x - 1/x$ substitutions.** The identity $x^2 + 1/x^2 = (x \pm 1/x)^2 \mp 2$ — the sign of the middle term depends on which substitution you choose.

4. **Assuming $f(x) = cx$ is the only Cauchy solution.** Over $\mathbb{R}$ without continuity, pathological solutions exist. Always state your assumptions.

5. **Dividing equations without checking for zero denominators.** In C3, dividing gives $x/y = 2$, but only if $xy + y^2 \neq 0$. Check the zero case separately.

"A formula you have derived yourself is worth a hundred you have memorised."

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths