Daily Algebra Drill #4 — Answers & Investigations

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Section	Topic	Questions	Target time
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Geometric series identity · Polynomial remainder shortcut · Parity and mod arguments · AM–GM under substitution.

Section A Rapid Recognition

1. Evaluate without expanding: 201×199.

Answer: 39,999

Method: $(200+1)(200-1) = 200^2 - 1 = 40000 - 1 = 39999$. DOTS with $a = 200$.

> **Investigate further:** This is Fermat's factorisation idea in reverse. Show that every product of two integers equidistant from n equals $n^2 - d^2$ where d is the distance. Use it to compute 1007×993 and 10001×9999 mentally.

2. Factorise: $x^2 - x - 42$.

Answer: $(x-7)(x+6)$

Method: Seek integers with product -42 and sum -1: that's -7 and $+6$.

> **Investigate further:** For $x^2 + bx + c$, the factorisation exists over $\mathbb{Z}$ iff $b^2 - 4c$ is a perfect square. Check: $1 + 168 = 169 = 13^2$. This discriminant test is always faster than trial and error.

3. Simplify $1 + x + x^2 + x^3 + x^4$ at $x = -1$.

Answer: 1

Method: Substitute: $1 - 1 + 1 - 1 + 1 = 1$. Alternatively, an odd number of terms of $1 + (-1) + \dots$ always gives 1 when the last term is $+1$.

> **Investigate further:** Geometric series: $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$ for $x \neq 1$. At $x = -1$, $n = 4$: $\frac{(-1)^5-1}{-2} = \frac{-2}{-2} = 1$. Explore the pattern for general even/odd n.

4. Given $p + q = 6$, $pq = 7$, find $p^3 + q^3$.

Answer: 90

Method: $p^3 + q^3 = (p+q)[(p+q)^2 - 3pq] = 6[36 - 21] = 6 \times 15 = 90$.

> **Investigate further:** Find $p^4 + q^4$ using the Newton recurrence $s_n = (p+q)s_{n-1} - pq \cdot s_{n-2}$: $s_4 = 6 \cdot s_3 - 7 \cdot s_2$. First find $s_2 = 36 - 14 = 22$, then $s_3 = 90$, then $s_4 = 6(90) - 7(22) = 540 - 154 = 386$.

5. Remainder when $f(x) = x^3 + 2x - 3$ is divided by $(x-1)$.

Answer: 0

Method: $f(1) = 1 + 2 - 3 = 0$. By the Remainder Theorem the remainder is $f(1) = 0$, so $(x - 1)$ is a factor.

> **Investigate further:** Since $(x-1)$ is a factor, fully factorise x^3+2x-3. Use synthetic or polynomial division: $x^3+2x-3 = (x-1)(x^2+x+3)$. Check that x^2+x+3 has negative discriminant, confirming $x = 1$ is the only real root.

6. Factorise completely: $2x^3 - 18x$.

Answer: $2x(x - 3)(x + 3)$

Method: Factor out $2x$: $2x(x^2 - 9) = 2x(x - 3)(x + 3)$. Extract scalars and common factors *first*, always.

> **Investigate further:** The zeros are $x = 0, \pm 3$. Sketch $y = 2x^3 - 18x$ by noting it passes through these zeros and has leading coefficient $+2$. Confirm the local max and min lie between consecutive zeros.

7. Simplify: $\dfrac{3^{n+1} - 3^{n-1}}{3^n}$.

Answer: $\dfrac{8}{3}$

Method: Factor out 3^{n-1}: $\dfrac{3^{n-1}(9 - 1)}{3^n} = \dfrac{8}{3}$. Always pull out the lowest power.

> **Investigate further:** Generalise: $\dfrac{a^{n+k} - a^{n-k}}{a^n} = a^k - a^{-k}$ for any base $a \neq 0$. Verify with $a = 2$, $k = 2$ and with $a = 10$, $k = 1$.

8. Given $a - b = 3$ and $a^2 + b^2 = 29$, find ab.

Answer: $ab = 10$

Method: $(a - b)^2 = a^2 - 2ab + b^2 = 9$, so $29 - 2ab = 9$, giving $ab = 10$.

> **Investigate further:** With $ab = 10$ and $a - b = 3$: form $x^2 - Sx + P = 0$ where $S = a + b = \pm\sqrt{(a - b)^2 + 4ab} = \pm\sqrt{49} = \pm 7$. Both signs are valid: $S = 7$ gives $(a, b) = (5, 2)$; $S = -7$ gives $(a, b) = (-2, -5)$. Check both satisfy the original data — dropping the $\pm$ silently loses a solution.

9. Factorise $x^4 - 1$ completely over $\mathbb{R}$.

Answer: $(x^2 + 1)(x + 1)(x - 1)$

Method: Two applications of DOTS: $(x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1)$. The factor $x^2 + 1$ is irreducible over $\mathbb{R}$.

> **Investigate further:** Over $\mathbb{C}$: $x^2 + 1 = (x + i)(x - i)$, giving four linear factors corresponding to the four fourth roots of unity: $\pm 1, \pm i$. Draw these on an Argand diagram and note they form a square — this is why $x^4 = 1$ has four roots equally spaced at $90°$.

10. Simplify: $\dfrac{n!}{(n - 2)!}$.

Answer: $n(n-1)$

Method: $\dfrac{n!}{(n-2)!} = n \cdot (n-1) \cdot \frac{(n-2)!}{(n-2)!} = n(n-1)$.

> **Investigate further:** This is the number of ways to choose an ordered pair from n items (permutations $P(n,2)$). More generally, $P(n,k) = \frac{n!}{(n-k)!}$. Verify $P(5,2) = 20$ by listing pairs from $\{1,2,3,4,5\}$.

Section B Manipulation Drills

1. Simplify $\dfrac{x^5-1}{x-1}$ using the geometric series formula.

Answer: $x^4 + x^3 + x^2 + x + 1$

Method: $x^5-1 = (x-1)(x^4+x^3+x^2+x+1)$. This is the finite geometric series: $1+x+\cdots+x^{n-1} = \frac{x^n-1}{x-1}$ with $n=5$.

> **Investigate further:** Prove the geometric series formula by letting $S = 1 + x + \cdots + x^{n-1}$ and computing $xS - S$. Then use it to sum $1 + 2 + 4 + \cdots + 2^9$ and $1 + 3 + 9 + \cdots + 3^7$.

2. Find the remainders when $g(x) = x^{10} - 3x^5 + 2$ is divided by $(x-1)$ and $(x+1)$.

Answer: Remainder at $(x-1)$: $g(1) = 1 - 3 + 2 = 0$. Remainder at $(x+1)$: $g(-1) = 1 + 3 + 2 = 6$.

Method: Remainder Theorem: evaluate g at the root of each linear factor. No division needed.

> **Investigate further:** Since $g(1) = 0$, the factor $(x-1)$ divides $g(x)$. Factorise $u^2 - 3u + 2 = (u-1)(u-2)$ where $u = x^5$, giving $g(x) = (x^5-1)(x^5-2) = (x-1)(x^4+x^3+x^2+x+1)(x^5-2)$. Check this against $g(1) = 0$ and $g(-1) = 6$.

3. Simplify: $\dfrac{ab}{a-b} - \dfrac{a^2b}{a^2-b^2}$.

Answer: $\dfrac{ab^2}{a^2-b^2}$

Method: Common denominator $(a-b)(a+b)$: numerator is $ab(a+b) - a^2b = ab\,[(a+b)-a] = ab^2$. Result: $\dfrac{ab^2}{a^2-b^2}$. Factor the shared ab out of the numerator before expanding anything.

> **Investigate further:** Practice combining algebraic fractions by always factoring denominators *before* finding a common denominator. Try $\dfrac{1}{x^2-1} - \dfrac{1}{x^2+2x+1}$ using this approach.

4. Factorise $x^6 - y^6$ completely over $\mathbb{Z}$.

Answer: $(x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$

Method: Two routes then combined: $x^6 - y^6 = (x^3-y^3)(x^3+y^3)$, then sum/difference of cubes on each factor. Alternatively $(x^2)^3 - (y^2)^3$ first, then DOTS on each.

> **Investigate further:** The fully factored form reveals that $x^6 - y^6$ is divisible by both $x^2 - y^2$ and $x^2 + xy + y^2$. How many distinct integer factors does $2^6 - 1^6 = 63$ have? List them using the

factorisation.

5. Roots of $3x^2 + 5x - 2 = 0$ are α, β. Find $\frac{1}{\alpha} + \frac{1}{\beta}$.

Answer: $\dfrac{5}{2}$

Method: $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-5/3}{-2/3} = \frac{5}{2}$.

Investigate further: The quadratic with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is obtained by replacing x with $\frac{1}{x}$: $\frac{3}{x^2} + \frac{5}{x} - 2 = 0 \Rightarrow 2x^2 - 5x - 3 = 0$. Verify $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{5}{2}$ from its Vieta formulas.

6. Simplify: $\left(1 - \dfrac{1}{x^2}\right)(x^2 + x)$.

Answer: $\dfrac{(x-1)(x+1)^2}{x}$

Method: $\frac{x^2-1}{x^2} \cdot x(x+1) = \frac{(x-1)(x+1)}{x^2} \cdot x(x+1) = \frac{(x-1)(x+1)^2}{x}$.

Investigate further: For what values of x is this expression zero, undefined, or equal to x? Setting it equal to x leads to the cubic $x^3 - x - 1 = 0$ — use the rational root theorem to prove it has *no* rational solutions. (Its one real root ≈ 1.3247 is the "plastic number".)

7. Given $a + b + c = 5$, $ab + bc + ca = 6$, find $a^2 + b^2 + c^2$.

Answer: 13

Method: $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \Rightarrow 25 = a^2 + b^2 + c^2 + 12 \Rightarrow a^2 + b^2 + c^2 = 13$.

Investigate further: Also find $a^3 + b^3 + c^3$ using Newton's identity $p_3 = e_1p_2 - e_2p_1 + 3e_3$ where $e_3 = abc$ is unknown. What additional condition would you need to pin down e_3?

8. Rearrange for y: $\dfrac{2y+1}{y-3} = x$.

Answer: $y = \dfrac{3x+1}{x-2}$

Method: $2y + 1 = x(y-3) = xy - 3x \Rightarrow 2y - xy = -3x - 1 \Rightarrow y(2-x) = -(3x+1) \Rightarrow y = \dfrac{3x+1}{x-2}$.

Investigate further: Notice that if $f(x) = \frac{2x+1}{x-3}$, then $f^{-1}(x) = \frac{3x+1}{x-2}$. Compute $f(f^{-1}(x))$ to verify. Then check: is $f(f(x)) = x$? If so, f is its own inverse (an *involution*).

9. Show $4n^2 - 1 = (2n-1)(2n+1)$; hence prove $4n^2 - 1$ is never prime for $n \geq 2$.

Answer: For $n \geq 2$: $2n - 1 \geq 3$ and $2n + 1 \geq 5$, both greater than 1, so $4n^2 - 1$ is a product of two integers each > 1. Hence composite.

Method: The key is that $(2n-1)$ and $(2n+1)$ are *both* greater than 1 for $n \geq 2$. For $n = 1$: $4 - 1 = 3$ which is prime, confirming the bound $n \geq 2$ is tight.

> **Investigate further:** This proves that no number of the form $4n^2 - 1$ (i.e. one less than a multiple of 4 that is a perfect square) is prime for $n \geq 2$. Extend: show $9n^2 - 1$ is composite for $n \geq 2$ and $p^2n^2 - 1$ is composite for any prime p and $n \geq 2$.

10. With $\alpha + \beta = 4$, $\alpha\beta = 1$, find the quadratic with integer coefficients whose roots are $\alpha - \frac{1}{\alpha}$ and $\beta - \frac{1}{\beta}$.

Answer: $x^2 - 12 = 0$

Method: Sum $= (\alpha+\beta) - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = 4 - \frac{\alpha+\beta}{\alpha\beta} = 4 - 4 = 0$. Product $= (\alpha - \frac{1}{\alpha})(\beta - \frac{1}{\beta}) = \alpha\beta - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} = 1 - \frac{\alpha^2+\beta^2}{\alpha\beta} + 1 = 2 - (16-2) = -12$. Quadratic: $x^2 - 0 \cdot x + (-12) = x^2 - 12$.

> **Investigate further:** The roots are $\pm 2\sqrt{3}$. Verify by solving $x^2 = 12$. Now try the same technique for the quadratic with roots $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$ — you should get a different quadratic. This technique of building new quadratics from transformed roots without solving is a core TMUA skill.

Section C Substitution & Structure

1. Solve: $4^x - 5 \cdot 2^x + 4 = 0$.

Answer: $x = 0$ and $x = 2$

Method: Let $u = 2^x$: $u^2 - 5u + 4 = (u-1)(u-4) = 0$. $u = 1 \Rightarrow x = 0$; $u = 4 \Rightarrow x = 2$.

> **Investigate further:** Now solve $4^x - 17 \cdot 2^x + 16 = 0$ and $9^x - 4 \cdot 3^x + 3 = 0$ by the same substitution. Then investigate: $a^{2x} - k \cdot a^x + (k-1) = 0$ always has $x = 0$ as a solution. What is the second solution?

2. Express $x^3 + \frac{1}{x^3}$ in terms of $t = x + \frac{1}{x}$.

Answer: $t^3 - 3t$

Method: $x^2 + \frac{1}{x^2} = t^2 - 2$. Then $x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right) = t(t^2 - 3) = t^3 - 3t$.

> **Investigate further:** This gives the chain: $s_1 = t$, $s_2 = t^2 - 2$, $s_3 = t^3 - 3t$, $s_4 = t^4 - 4t^2 + 2$, following the Newton recurrence $s_n = t \cdot s_{n-1} - s_{n-2}$. Derive s_4 and s_5, then evaluate each at $t = 3$.

3. Find the minimum of $3x + \frac{27}{x}$ for $x > 0$.

Answer: Minimum is 18 at $x = 3$.

Method: AM–GM: $3x + \frac{27}{x} \geq 2\sqrt{3x \cdot \frac{27}{x}} = 2\sqrt{81} = 18$. Equality when $3x = \frac{27}{x} \Rightarrow x^2 = 9 \Rightarrow x = 3$.

> **Investigate further:** Compare to completing the square: $3x + \frac{27}{x} = 3\left(\sqrt{x} - \frac{3}{\sqrt{x}}\right)^2 + 18 \geq 18$. Both proofs work; which is faster? Then find the minimum of $ax + \frac{b}{x}$ for $a, b, x > 0$. (Answer: $2\sqrt{ab}$ at $x = \sqrt{b/a}$.) Apply to minimise $5x + \frac{20}{x}$.

4. Solve for real x: $\sqrt{2x-1} + \sqrt{x-1} = 2$.

Answer: $x = 12 - 4\sqrt{7}$ (one real solution)

Method: Isolate $\sqrt{2x-1} = 2 - \sqrt{x-1}$, requiring $2 \geq \sqrt{x-1}$, i.e. $x \leq 5$. Square: $2x - 1 = 4 - 4\sqrt{x-1} + (x-1)$, so $x - 4 = -4\sqrt{x-1}$, requiring $x \leq 4$. Square again: $(x-4)^2 = 16(x-1) \Rightarrow$

$x^2 - 24x + 32 = 0 \Rightarrow x = 12 \pm 4\sqrt{7}$. Since $x \le 4$: $x = 12 - 4\sqrt{7} \approx 1.42$.

> **Investigate further:** For cleaner surd equations, try the conjugate trick: also form $\sqrt{2x-1} - \sqrt{x-1}$ by multiplying through by $\frac{\sqrt{2x-1}-\sqrt{x-1}}{\sqrt{2x-1}-\sqrt{x-1}}$. This gives a second equation and the system solves more cleanly.

5. Find the value of k such that $(x+1)(x+3)(x+5)(x+7)+k$ is a perfect square.

Answer: $k = 16$

Method: Pair: $(x+1)(x+7) = x^2+8x+7$ and $(x+3)(x+5) = x^2+8x+15$. Let $u = x^2+8x+11$: product $= (u-4)(u+4) = u^2 - 16$. This is a perfect square u^2 when $k = 16$.

> **Investigate further:** The same pairing idea works for symmetric arithmetic progressions. Try proving: $(n-3)(n-1)(n+1)(n+3)+16 = (n^2-5)^2$. Then generalise to $\{n-3d, n-d, n+d, n+3d\}$ and find the constant that completes a square.

6. Prove $(a^2+b^2)(c^2+d^2) \ge (ac+bd)^2$. State the equality condition.

Answer: Equality iff $ad = bc$.

Method: $(a^2+b^2)(c^2+d^2) - (ac+bd)^2 = a^2d^2 - 2abcd + b^2c^2 = (ad-bc)^2 \ge 0$. $\square$ This is the Cauchy–Schwarz inequality in two dimensions.

> **Investigate further:** The Cauchy–Schwarz inequality in n dimensions states $(\mathbf{u}\cdot\mathbf{v})^2 \le |\mathbf{u}|^2|\mathbf{v}|^2$, which in components is $(\sum a_ib_i)^2 \le (\sum a_i^2)(\sum b_i^2)$. Prove the $n = 3$ case by expanding $(a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2 \ge 0$.

7. With $u = x+y$, $v = x-y$, express x^4+y^4 in terms of u and v.

Answer: $\dfrac{u^4+v^4+6u^2v^2}{8}$

Method: $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$. $x^4+y^4 = \frac{1}{16}\left[(u+v)^4 + (u-v)^4\right] = \frac{1}{16}\cdot 2(u^4+6u^2v^2+v^4) = \frac{u^4+6u^2v^2+v^4}{8}$.

> **Investigate further:** This substitution ($u = x+y$, $v = x-y$) is called a *linear change of variables.* It converts symmetric expressions in x, y into expressions in u, v where even-degree terms survive. Use it to express x^6+y^6 in terms of u and v.

8. Solve: $x^4 - 7x^2 + 12 = 0$.

Answer: $x = \pm 2,\ \pm\sqrt{3}$

Method: Let $u = x^2$: $(u-3)(u-4) = 0$. $u = 3 \Rightarrow x = \pm\sqrt{3}$; $u = 4 \Rightarrow x = \pm 2$.

> **Investigate further:** Sketch $y = x^4 - 7x^2 + 12$ by noting the zeros and that $y \to +\infty$ as $x \to \pm\infty$. Find the local minima by writing $y = (x^2 - \frac{7}{2})^2 - \frac{1}{4}$ — what are the y-values at the minima?

Section D Challenge (TMUA / SMC difficulty)

1. Prove $n^2 \equiv 0$ or $1 \pmod 4$ for all integers n; hence n^2+2 is never divisible by 4.

Answer: Proved below.

Method: Every integer is even ($n = 2m$) or odd ($n = 2m+1$). *Even:* $n^2 = 4m^2 \equiv 0$. *Odd:* $n^2 = 4m^2+4m+1 \equiv 1$. So $n^2 \equiv 0$ or 1, hence $n^2+2 \equiv 2$ or $3 \pmod 4$. Neither is $\equiv 0$, so $4 \nmid (n^2+2)$. $\square$

> **Investigate further:** Extend: show $n^2 \equiv 0, 1, \text{ or } 4 \pmod 8$ by checking all residues $n \equiv 0, \ldots, 7$. Hence show that $n^2 + 3$ is never divisible by 8, and that $n^2 + n$ is always even.

2. Find all real solutions to $x^4 - 4x^3 + 4x^2 - 1 = 0$.

Answer: $x = 1$ (double) and $x = 1 \pm \sqrt{2}$

Method: Try $(x^2 + ax + b)(x^2 + cx + d)$ with $b + d + ac = 4$, $bd = -1$, $a + c = -4$, $ad + bc = 0$. Taking $b = 1$, $d = -1$: $a = c = -2$ works. So $(x^2 - 2x + 1)(x^2 - 2x - 1) = 0$, giving $(x - 1)^2 = 0 \Rightarrow x = 1$ and $x^2 - 2x - 1 = 0 \Rightarrow x = 1 \pm \sqrt{2}$.

> **Investigate further:** Notice that this quartic has the form $q(x^2 - 2x)$ where $q(t) = t^2 - 1$. This substitution-based structure ($t = x^2 - 2x = (x - 1)^2 - 1$) is a powerful pattern. Find all solutions to $x^4 - 4x^3 + 4x^2 = 9$ using the same substitution.

3. Sequence: $a_1 = 1$, $a_{n+1} = \frac{a_n}{a_n + 2}$. Find a closed form for a_n and a_{100}.

Answer: $a_n = \dfrac{1}{2^n - 1}; \quad a_{100} = \dfrac{1}{2^{100} - 1}$

Method: Let $b_n = \frac{1}{a_n}$: $b_{n+1} = \frac{a_n + 2}{a_n} = 1 + \frac{2}{a_n} = 1 + 2b_n$. This linear recurrence has homogeneous solution $C \cdot 2^n$ and particular solution -1, giving $b_n = C \cdot 2^n - 1$. With $b_1 = 1$: $C = 1$. So $b_n = 2^n - 1$ and $a_n = \frac{1}{2^n - 1}$.

> **Investigate further:** The substitution $b_n = 1/a_n$ converted a nonlinear recurrence to a linear one — a standard trick. Now solve $a_{n+1} = \frac{2a_n}{a_n + 3}$ with $a_1 = 1$ by the same method. Note that as $n \to \infty$, $a_n \to 0$: the sequence converges to 0.

4. Evaluate $\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right)$ in closed form. Find its limit as $n \to \infty$.

Answer: $\dfrac{n + 1}{2n}; \quad \text{limit} = \dfrac{1}{2}$

Method: $1 - \frac{1}{k^2} = \frac{(k-1)(k+1)}{k^2}$. Product $= \dfrac{\prod(k - 1)}{\prod k} \cdot \dfrac{\prod(k + 1)}{\prod k} = \frac{1}{n} \cdot \frac{n+1}{2} = \frac{n+1}{2n}$. Each factor telescopes separately: $\prod_{k=2}^{n} \frac{k-1}{k} = \frac{1}{n}$ and $\prod_{k=2}^{n} \frac{k+1}{k} = \frac{n+1}{2}$.

> **Investigate further:** The infinite product $\prod_{k=2}^{\infty}(1 - \frac{1}{k^2}) = \frac{1}{2}$ has a beautiful proof via the Weierstrass product for $\sin(\pi x)$: $\sin(\pi x) = \pi x \prod_{k=1}^{\infty}(1 - \frac{x^2}{k^2})$. Setting $x = 1$ gives $0 = \pi \cdot 0$ (trivial). Setting $x = \frac{1}{2}$: $1 = \frac{\pi}{2} \prod_{k=1}^{\infty}(1 - \frac{1}{4k^2})$. Explore the Wallis product formula for π.

5. $p(x) = x^4 + ax^3 + bx^2 + cx + d$ with $p(1) = 10$, $p(2) = 20$, $p(3) = 30$, $p(4) = 40$. Find $p(12) - 12p(0)$.

Answer: 7752

Method: Define $q(x) = p(x) - 10x$. Then $q(1) = q(2) = q(3) = q(4) = 0$. Since p is monic of degree 4: $q(x) = (x - 1)(x - 2)(x - 3)(x - 4)$. $p(0) = q(0) = (-1)(-2)(-3)(-4) = 24$. $p(12) = q(12) + 120 = (11)(10)(9)(8) + 120 = 7920 + 120 = 8040$. $p(12) - 12p(0) = 8040 - 288 = 7752$.

> **Investigate further:** The key idea: defining $q(x) = p(x) - f(x)$ where f is simple and matches the given values reduces the problem to finding q, which has known zeros. Explore: if $p(k) = k^2$ for $k = 0, 1, 2, 3, 4$ where p is monic of degree 5, find $p(5)$. (Define $q(x) = p(x) - x^2$ and proceed.)

Top 5 Patterns — Worksheet #4

1. **Geometric series factorisation.** $x^n - 1 = (x-1)(x^{n-1} + \cdots + 1)$. Before dividing any polynomial by $(x-1)$, check if the Remainder Theorem gives 0 first.
2. **AM–GM for expressions of the form $f(x) + c/f(x)$.** Minimum $= 2\sqrt{c}$ at $f(x) = \sqrt{c}$. Recognise the pattern immediately.
3. **Remainder Theorem over long division.** Always evaluate $f(a)$ instead of dividing by $(x-a)$.
4. **Mod-4 parity argument.** $n^2 \equiv 0$ or $1 \pmod 4$ for all n. Memorise this — it rules out divisibility by 4 for many expressions.
5. **Polynomial masking: $q(x) = p(x) - f(x)$.** When $p(k) = f(k)$ at several points, define $q = p - f$ to get a polynomial with known zeros.

Common Traps — Worksheet #4

1. **Forgetting to check domain when squaring.** In C4, squaring introduces extraneous solutions. Always substitute back.
2. **Confusing $x^4 - 1$ factors.** Over $\mathbb{R}$: three factors. Over $\mathbb{C}$: four. State your field.
3. **AM–GM requires positive terms.** $3x + 27/x \geq 18$ only for $x > 0$. Check sign before applying.
4. **Missing the polynomial masking trick.** In D5, trying to solve for a, b, c, d directly from three equations is impossible (underdetermined). Always look for a cleaner reformulation.
5. **Partial telescoping errors.** In D4, separate the product into two telescoping products before computing. Do not try to cancel within a single combined fraction.

"The person who can solve in 5 minutes what others solve in 50 has not worked 10 times harder — they have seen something different."

Found a mistake or want to contribute a sheet?
Visit the open-source project at github.com/The-CerealDev/speed-maths