# Daily Algebra Drill #3 — Answers & Investigations

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| Section | Topic | Questions | Target time |
|---|---|---|---|
| A | Rapid Recognition | 10 | 2 min 30 sec |
| B | Manipulation Drills | 10 | 8 min |
| C | Substitution & Structure | 8 | 10 min |
| D | Challenge | 5 | 15 min |

*New toolkit today: Remainder & Factor Theorem · AM–GM & algebraic inequalities · Completing the square in two variables · Rational Root Theorem · Modular/divisibility arguments · Parametric families of solutions.*

## Section A Rapid Recognition

**1.** $f(x) = x^3 - 3x^2 + kx - 8$ has $(x-2)$ as a factor. Find $k$.

**Answer:** $k = 6$

**Method: Factor Theorem:** if $(x-2)$ is a factor then $f(2) = 0$. $8 - 12 + 2k - 8 = 0 \Rightarrow 2k = 12 \Rightarrow k = 6$. Never do polynomial long division when the factor theorem gives a one-line equation.

> **Investigate further: Remainder Theorem (full statement):** when $p(x)$ is divided by $(x-a)$, the remainder is $p(a)$. Use this to find the remainder when $f(x) = x^3 - 6x^2 + 11x - 6$ is divided by $(x-4)$. Then explore: if the remainder when $f(x)$ is divided by $(x-1)$ is 3 and by $(x+1)$ is $-1$, find the remainder when divided by $(x^2-1)$. (*Key: the remainder is now linear: $ax+b$. Set up two equations.*)

**2.** Find the remainder when $p(x) = x^{100} - 1$ is divided by $(x-1)$.

**Answer:** 0

**Method:** $p(1) = 1^{100} - 1 = 0$. So the remainder is 0 and $(x-1)$ is in fact a factor. The Remainder Theorem makes this instantaneous — no division required.

> **Investigate further:** Find the remainder when $x^{100} + x^{50} + 1$ is divided by $(x-1)$, $(x+1)$, and $(x^2-1)$. Then prove: $x^n - 1$ is always divisible by $(x-1)$ for positive integers $n$. (*Use the factorisation $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + 1)$, which you can verify by multiplying out.*)

**3.** Factorise: $6x^2 - x - 12$.

**Answer:** $(2x-3)(3x+4)$

**Method:** Product $= 6 \times (-12) = -72$. Seek pair summing to $-1$: $-9$ and 8. Split: $6x^2 - 9x + 8x - 12 = 3x(2x-3) + 4(2x-3) = (2x-3)(3x+4)$.

> **Investigate further: AC method / discriminant connection.** A quadratic $ax^2+bx+c$ factorises over $\mathbb{Z}$ if and only if its discriminant $b^2 - 4ac$ is a perfect square. Check: $(-1)^2 - 4(6)(-12) = 1 + 288 = 289 = 17^2$. ✓This gives the roots directly: $x = \frac{1 \pm 17}{12}$, i.e. $x = \frac{3}{2}$ or $x = -\frac{4}{3}$. Investigate: for which integer values of $c$ does $6x^2 - x + c$ factorise over $\mathbb{Z}$?

**4.** Simplify: $\dfrac{x^{2n} - 1}{x^n - 1}$.

**Answer:** $x^n + 1$

**Method:** Difference of two squares: $x^{2n} - 1 = (x^n - 1)(x^n + 1)$. Cancel $(x^n - 1)$. This is the $m = 2$ case of the general factorisation $x^{mn} - 1$.

> **Investigate further:** Generalise: $\dfrac{x^{mn} - 1}{x^n - 1} = x^{n(m-1)} + x^{n(m-2)} + \cdots + x^n + 1$, a geometric series in $x^n$. Use this with $x = 10$, $n = 1$ to explain why $\underbrace{99\cdots9}_{m}$ is always divisible by 9. Then use $n = 2$, $m = 3$ to factorise $x^6 - 1$ fully over $\mathbb{Z}$.

**5.** Given $x > 0$, find the minimum value of $x + \dfrac{9}{x}$.

**Answer:** 6

**Method:** AM–GM: $x + \dfrac{9}{x} \geq 2\sqrt{x \cdot \dfrac{9}{x}} = 2 \cdot 3 = 6$. Equality when $x = \frac{9}{x}$, i.e. $x = 3$. Always suspect AM–GM when you see a sum of a quantity and its reciprocal (scaled).

> **Investigate further:** Generalise: the minimum of $x + \dfrac{c}{x}$ for $x, c > 0$ is $2\sqrt{c}$, achieved at $x = \sqrt{c}$. Now find the minimum of $4x + \dfrac{9}{x}$ for $x > 0$, and in general of $ax + \dfrac{c}{x}$ for $a, c > 0$. (*Answer:* $2\sqrt{ac}$, *at* $x = \sqrt{c/a}$.)

**6.** Factorise over $\mathbb{Z}$: $x^4 - y^4$.

**Answer:** $(x^2 + y^2)(x + y)(x - y)$

**Method:** Apply difference of two squares twice: $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$. The factor $x^2 + y^2$ does not factorise further over $\mathbb{R}$.

> **Investigate further:** Over $\mathbb{C}$, $x^2 + y^2 = (x + iy)(x - iy)$. So $x^4 - y^4 = (x + iy)(x - iy)(x + y)(x - y)$ over $\mathbb{C}$. Now factorise $x^4 - 1$ fully over $\mathbb{R}$, then over $\mathbb{C}$. Connect to the *roots of unity*: the four roots of $x^4 = 1$ are $1, -1, i, -i$, corresponding exactly to the four factors.

**7.** The roots of $2x^2 - 6x + 3 = 0$ are $\alpha$ and $\beta$. Find $\alpha^2 + \beta^2$.

**Answer:** 6

**Method:** Vieta: $\alpha + \beta = 3$, $\alpha\beta = \frac{3}{2}$. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 9 - 3 = 6$. Note: for a non-monic quadratic $ax^2 + bx + c = 0$, Vieta gives $\alpha + \beta = -b/a$, $\alpha\beta = c/a$.

> **Investigate further:** Find $\alpha^3 + \beta^3$ and $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$ using the same roots. For $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$, write it as $\dfrac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$. Then investigate: what quadratic has roots $\alpha^2$ and $\beta^2$? (Use the Vieta data for the new quadratic: sum $= \alpha^2 + \beta^2$, product $= \alpha^2\beta^2$.)

**8.** Simplify: $\dfrac{(n+1)! - n!}{(n-1)!}$.

**Answer:** $n^2$

**Method:** Factor: $(n+1)! - n! = n! \cdot n = n \cdot n!$. Then $\dfrac{n \cdot n!}{(n-1)!} = n \cdot \dfrac{n!}{(n-1)!} = n \cdot n = n^2$. Always factor out the smaller factorial first.

> **Investigate further:** Prove the identity $(n+1)! - n! - n \cdot (n-1)! = (n-1)! \cdot (n^2 - n)$ by factoring $(n-1)!$ throughout. Then investigate: for which $n$ is $\frac{(2n)!}{(n!)^2}$ (the central binomial coefficient) divisible by $n+1$? (*Catalan number connection.*)

**9.** Show that $(a+b+c)^2 - (a^2+b^2+c^2) = 2(ab+bc+ca)$ and state $ab+bc+ca$ when $a+b+c=4$, $a^2+b^2+c^2=8$.

**Answer:** $ab+bc+ca=4$

**Method:** Expand $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$. Rearrange. Substituting: $16-8 = 2(ab+bc+ca) \Rightarrow ab+bc+ca = 4$.

> **Investigate further:** With $a+b+c=4$, $a^2+b^2+c^2=8$, $ab+bc+ca=4$: now use Newton's identity to find $a^3+b^3+c^3$ in terms of the elementary symmetric polynomials $e_1=4$, $e_2=4$, $e_3=abc$. Show that $a^3+b^3+c^3 = 3e_3 + e_1(e_1^2 - 3e_2) = 3abc + 16$. Hence $a^3+b^3+c^3$ depends on $abc$ — find specific triples $(a,b,c)$ satisfying the constraints and verify.

**10.** Find the value(s) of $k$ such that $x^2+kx+16$ is a perfect square.

**Answer:** $k = \pm 8$

**Method:** A perfect-square quadratic has discriminant zero: $k^2 - 64 = 0 \Rightarrow k = \pm 8$. Then $x^2+8x+16 = (x+4)^2$ and $x^2-8x+16 = (x-4)^2$.

> **Investigate further:** Generalise: $x^2+kx+c$ is a perfect square iff $k = \pm 2\sqrt{c}$. For $c$ to allow integer $k$, we need $c$ to be a perfect square itself. Now: for which integer values of $m$ is $x^2+mx+m$ a perfect square (i.e. has discriminant zero)? Find all such $m$.

## Section B Manipulation Drills

**1.** Find all rational roots of $2x^3 - 3x^2 - 11x + 6$ and factorise completely.

**Answer:** $(x-3)(x+2)(2x-1)$; rational roots: $x = 3,\ -2,\ \frac{1}{2}$

**Method: Rational Root Theorem:** rational roots have the form $\pm\frac{p}{q}$ where $p \mid 6$ and $q \mid 2$. Candidates: $\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}$. Test $x=3$: $54-27-33+6=0$. ✓ Divide out $(x-3)$ to get $2x^2+3x-2 = (2x-1)(x+2)$.

> **Investigate further: Rational Root Theorem (formal statement):** if $a_n x^n + \cdots + a_0$ has a rational root $p/q$ in lowest terms, then $p \mid a_0$ and $q \mid a_n$. Use this to show that $x^3 - 2$ has no rational roots, hence $\sqrt[3]{2}$ is irrational. Generalise: prove that $\sqrt[n]{k}$ is irrational whenever $k$ is not a perfect $n$th power.

**2.** Simplify: $\dfrac{x^3+1}{x^2-x+1}$.

**Answer:** $x+1$

**Method:** Sum of cubes: $x^3+1 = (x+1)(x^2-x+1)$. Cancel $(x^2-x+1)$. Check: $x^2-x+1 \neq 0$ for real $x$ (discriminant $< 0$).

> **Investigate further:** The factor $x^2-x+1$ is the *cyclotomic polynomial* $\Phi_6(x)$. Its roots are the

> primitive 6th roots of unity: $e^{\pm i\pi/3}$. Show that $x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$ by factorising step by step. This connects to the factorisation of $x^n - 1$ into cyclotomic polynomials, a key tool in number theory and algebra.

**3.** Given $a+b=p$ and $a^3+b^3=q$, express $ab$ in terms of $p$ and $q$.

**Answer:** $ab = \dfrac{p^3-q}{3p}$

**Method:** $a^3+b^3 = (a+b)(a^2-ab+b^2) = (a+b)\big[(a+b)^2 - 3ab\big] = p(p^2-3ab) = q$. So $p^3 - 3p\cdot ab = q$, giving $ab = \dfrac{p^3-q}{3p}$.

> **Investigate further:** Given only $a+b=p$ and $a^3+b^3=q$, find $a^4+b^4$ and $a^5+b^5$ using Newton's recurrence $s_n = (a+b)s_{n-1} - ab\cdot s_{n-2}$. Then: if $p$ and $q$ are both integers, is $ab$ necessarily rational? Always? Find an example where $p, q \in \mathbb{Z}$ but $a, b \notin \mathbb{Q}$.

**4.** Simplify: $\dfrac{x^2(x+1) - x(x+1)^2}{x(x+1)}$.

**Answer:** $-1$

**Method:** Factor the numerator: $x(x+1)[x-(x+1)] = x(x+1)(-1)$. Cancel $x(x+1)$. Result: $-1$. Always look for a common factor in the numerator before combining fractions.

> **Investigate further:** This expression equals $-1$ for all $x \neq 0, -1$. What does this mean geometrically? The function $f(x) = \dfrac{x^2(x+1) - x(x+1)^2}{x(x+1)}$ is a rational function that simplifies to the constant $-1$ with two holes. Sketch it and compare to $y=-1$. Then investigate: when does a rational function simplify to a polynomial? What algebraic condition must hold?

**5.** Factorise: $a^3 - a^2b - ab^2 + b^3$.

**Answer:** $(a-b)^2(a+b)$

**Method:** Group: $(a^3+b^3) - (a^2b+ab^2) = (a+b)(a^2-ab+b^2) - ab(a+b) = (a+b)(a^2-2ab+b^2) = (a+b)(a-b)^2$.

> **Investigate further:** Verify using the substitution $b=a$: the expression becomes 0, confirming $(a-b)$ is a factor. Then check $b=-a$: again 0, confirming $(a+b)$ is a factor. Since the polynomial is degree 3 with two linear factors, the third must also be linear — and we can read it off. This substitution method ("*plug in values to find factors*") is the heart of the factor theorem. Use it to factorise $a^4 - b^4 - a^3b + ab^3$.

**6.** Use $\displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ to find $\displaystyle\sum_{r=1}^{n}(2r-1)$.

**Answer:** $n^2$

**Method:** $\displaystyle\sum_{r=1}^{n}(2r-1) = 2\cdot\frac{n(n+1)}{2} - n = n(n+1) - n = n^2$. The sum of the first $n$ odd numbers is always a perfect square. Elegant!

> **Investigate further:** Prove the same result geometrically: the $n$th odd number $2n-1$ is the number of extra unit squares added to an $n \times n$ grid to produce an $(n+1) \times (n+1)$ grid (the L-shaped "gnomon"). Then find $\sum_{r=1}^{n}(2r)$ and $\sum_{r=1}^{n} r^2$ using the identity $(r+1)^3 - r^3 = 3r^2 + 3r + 1$, telescoped.

**7.** Solve: $\dfrac{2x-1}{x+3} - \dfrac{x+2}{x-1} = 1$.

**Answer:** $x = -\dfrac{1}{5}$

**Method:** Common denominator $(x+3)(x-1)$: $(2x-1)(x-1) - (x+2)(x+3) = (x+3)(x-1)$. LHS: $(2x^2 - 3x + 1) - (x^2 + 5x + 6) = x^2 - 8x - 5$. RHS: $x^2 + 2x - 3$. So $x^2 - 8x - 5 = x^2 + 2x - 3 \Rightarrow -10x = 2 \Rightarrow x = -\frac{1}{5}$. Check in the original: LHS $= \frac{-7/5}{14/5} - \frac{9/5}{-6/5} = -\frac{1}{2} + \frac{3}{2} = 1$. ✓

> **Investigate further:** Rational equations of this form always reduce to a polynomial equation after clearing denominators. Investigate: under what conditions does clearing denominators in $\dfrac{P(x)}{Q(x)} = \dfrac{R(x)}{S(x)}$ produce extraneous solutions? (*Extraneous solutions satisfy the cleared equation but make a denominator zero.*) Give an example of an equation that gains an extraneous solution when cleared.

**8.** Simplify: $\left(\dfrac{1}{a} - \dfrac{1}{b}\right) \div \left(\dfrac{1}{a^2} - \dfrac{1}{b^2}\right)$.

**Answer:** $\dfrac{ab}{a+b}$

**Method:** Numerator: $\dfrac{b-a}{ab}$. Denominator: $\dfrac{b^2-a^2}{a^2b^2} = \dfrac{(b-a)(b+a)}{a^2b^2}$. Dividing: $\dfrac{b-a}{ab} \cdot \dfrac{a^2b^2}{(b-a)(b+a)} = \dfrac{ab}{a+b}$.

> **Investigate further:** Note that $\dfrac{ab}{a+b} = \dfrac{1}{\frac{1}{a} + \frac{1}{b}}$, the *half-harmonic mean.* The harmonic mean of $a$ and $b$ is $H = \dfrac{2ab}{a+b}$. Prove the chain of inequalities $H \le G \le A$ where $G = \sqrt{ab}$ and $A = \frac{a+b}{2}$ for $a, b > 0$. (*Use AM–GM twice: once to get $G \le A$, once rearranged to get $H \le G$.*)

**9.** Show that $n^3 - n$ is divisible by 6 for all integers $n$.

**Answer:** Divisible by 6 for all $n \in \mathbb{Z}$.

**Method:** $n^3 - n = n(n-1)(n+1) = (n-1)n(n+1)$: the product of three consecutive integers. *Divisible by 2:* among any two consecutive integers, one is even. *Divisible by 3:* among any three consecutive integers, one is divisible by 3. Since $\gcd(2,3) = 1$, the product is divisible by 6. □

> **Investigate further: Generalisation.** Show that the product of any $k$ consecutive integers is divisible by $k!$. This is equivalent to showing that $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$ is always an integer for $n \ge k$. (*Proof: $\binom{n}{k}$ counts the number of $k$-element subsets of an $n$-element set, hence is an integer.*) Apply: show $n^5 - n$ is always divisible by 30 by factoring $n^5 - n = n(n^4 - 1)$ further.

**10.** Roots of $x^2 - x - 1 = 0$ are $\alpha, \beta$. Find $\alpha^3 + \beta^3$ and $\alpha^4 + \beta^4$.

**Answer:** $\alpha^3 + \beta^3 = 4; \quad \alpha^4 + \beta^4 = 7$

**Method:** Vieta: $\alpha + \beta = 1$, $\alpha\beta = -1$. Newton recurrence $s_n = (\alpha + \beta)s_{n-1} - \alpha\beta \cdot s_{n-2} = s_{n-1} + s_{n-2}$. $s_1 = 1$, $s_2 = (\alpha + \beta)^2 - 2\alpha\beta = 1 + 2 = 3$. $s_3 = s_2 + s_1 = 4$. $s_4 = s_3 + s_2 = 7$. This recurrence is the *Lucas sequence* — the roots are the golden ratio and its conjugate.

> **Investigate further: Golden ratio and Fibonacci.** The roots of $x^2 - x - 1 = 0$ are $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$ (the golden ratio and its conjugate). The power sums $s_n = \phi^n + \hat{\phi}^n$ satisfy $s_n = s_{n-1} + s_{n-2}$ (Fibonacci-like). Compute $s_0$ through $s_8$ and compare to the Lucas numbers: $2, 1, 3, 4, 7, 11, 18, 29, 47$. Prove that $\phi^n = F_n\phi + F_{n-1}$ where $F_n$ is the $n$th Fibonacci number.

## Section C Substitution & Structure

**1.** Solve: $9^x - 10 \cdot 3^x + 9 = 0$.

**Answer:** $x = 0$ and $x = 2$

**Method:** Let $u = 3^x$: $u^2 - 10u + 9 = 0 \Rightarrow (u-1)(u-9) = 0$. $u = 1 \Rightarrow 3^x = 1 \Rightarrow x = 0$. $u = 9 \Rightarrow 3^x = 9 \Rightarrow x = 2$.

> **Investigate further:** Solve $4^x - 5 \cdot 2^x + 4 = 0$ and $25^x - 26 \cdot 5^x + 25 = 0$ by the same method. Then investigate: $2^{2x} - k \cdot 2^x + k - 1 = 0$. Show that $x = 0$ is always a solution regardless of $k$, and find the second solution in terms of $k$. What happens when $k = 2$?

**2.** Find the minimum value of $x^2 + y^2$ subject to $x + 2y = 5$.

**Answer:** 5

**Method:** Substitute $x = 5-2y$: minimise $(5-2y)^2+y^2 = 5y^2-20y+25$. Complete the square: $5(y-2)^2+5$. Minimum is **5** at $y = 2$, $x = 1$. Alternatively (elegant): by Cauchy–Schwarz, $(x+2y)^2 \le (x^2+y^2)(1^2+2^2)$, so $x^2 + y^2 \ge \frac{25}{5} = 5$.

> **Investigate further: Cauchy–Schwarz inequality:** $(a_1b_1 + a_2b_2)^2 \le (a_1^2 + a_2^2)(b_1^2 + b_2^2)$. Prove it by expanding $(a_1b_2 - a_2b_1)^2 \ge 0$. Use it to find the minimum of $x^2 + y^2 + z^2$ subject to $x + y + 2z = 6$. Then connect to the geometric interpretation: $\sqrt{x^2 + y^2}$ is the distance from the origin to the line $x + 2y = 5$, and the minimum is the perpendicular distance.

**3.** Prove that $a^2 + b^2 + c^2 \ge ab + bc + ca$ for all real $a, b, c$.

**Answer:** Proof: see method.

**Method:** $2(a^2 + b^2 + c^2) - 2(ab + bc + ca) = (a-b)^2 + (b-c)^2 + (c-a)^2 \ge 0$. Divide by 2. Equality iff $a = b = c$. $\square$ This is the most important algebraic inequality after AM–GM.

> **Investigate further:** The expression $(a-b)^2 + (b-c)^2 + (c-a)^2$ is a *sum of squares (SOS) decomposition*, which proves non-negativity constructively. Now prove: $a^4+b^4+c^4 \ge a^2b^2+b^2c^2+c^2a^2$ using the same idea (replace $a, b, c$ with $a^2, b^2, c^2$). Then investigate: is $a^2 + b^2 + c^2 \ge ab + bc + ca$ still true if $a, b, c$ are complex numbers? (*No — consider what '$\ge$' means for complex numbers.*)

**4.** Solve: $\sqrt{x+5} + \sqrt{x-3} = 4$.

**Answer:** $x = 4$

**Method:** Isolate one radical: $\sqrt{x+5} = 4 - \sqrt{x-3}$. Square: $x + 5 = 16 - 8\sqrt{x-3} + (x-3) \Rightarrow 8\sqrt{x-3} = 8 \Rightarrow x - 3 = 1 \Rightarrow x = 4$. Check: $\sqrt{9} + \sqrt{1} = 3 + 1 = 4$. ✓

> **Investigate further:** The conjugate trick: instead of isolating, multiply both sides by $(\sqrt{x+5} - \sqrt{x-3})$. LHS becomes $(x+5)-(x-3)=8$, giving $\sqrt{x+5}-\sqrt{x-3}=2$. Combined with the original equation: $2\sqrt{x+5}=6 \Rightarrow x+5=9 \Rightarrow x=4$. This conjugate approach is faster and avoids squaring. Use it to solve $\sqrt{x+4}-\sqrt{x-4}=\dfrac{8}{5}$.

**5.** Find the minimum of $x^2+4x+y^2-6y+13$, and the $(x,y)$ at which it occurs.

**Answer:** Minimum is **0**, at $(x,y)=(-2,\,3)$

**Method:** Complete the square in each variable separately: $(x+2)^2-4+(y-3)^2-9+13=(x+2)^2+(y-3)^2 \geq 0$. Minimum is 0 when $x=-2$, $y=3$.

> **Investigate further:** Note $(x+2)^2+(y-3)^2=0$ means the expression equals the square of the distance from $(-2,3)$ to $(x,y)$. So the minimum is the point closest to the origin — but the expression vanishes *at* $(-2,3)$, not at the origin. Now find the minimum of $x^2+y^2+z^2-2x+4y-6z+14$. Is the minimum always achievable for quadratics in $n$ variables? (*Yes, iff the quadratic form is positive definite — a key idea in multivariable optimisation.*)

**6.** If $x+\dfrac{1}{x}=5$, find $\dfrac{x^4+1}{x^2}$.

**Answer:** 23

**Method:** $\dfrac{x^4+1}{x^2}=x^2+\dfrac{1}{x^2}=\left(x+\dfrac{1}{x}\right)^2-2=25-2=23.$

> **Investigate further:** Extend: find $\dfrac{x^6+1}{x^3}$ and $\dfrac{x^8+1}{x^4}$ using the chain: $s_n=\left(x+\frac{1}{x}\right)s_{n-1}-s_{n-2}$ where $s_k=x^k+\frac{1}{x^k}$. This is the same Newton recurrence as before, applied to $x$ and $\frac{1}{x}$ (whose sum is 5 and product is 1). Compute $s_1$ through $s_5$.

**7.** Solve over $\mathbb{R}$: $x^4+4x^2-5=0$.

**Answer:** $x=\pm 1$

**Method:** Let $u=x^2$: $(u+5)(u-1)=0$. $u=-5$ gives no real solutions. $u=1 \Rightarrow x=\pm 1$.

> **Investigate further:** For which values of $k$ does $x^4+4x^2+k=0$ have: (i) no real solutions, (ii) exactly two real solutions, (iii) four real solutions? (*Set $u=x^2$: the quadratic $u^2+4u+k=0$ has roots that must be positive.*) Sketch the curve $y=x^4+4x^2$ and read off the answers geometrically.

**8.** Given $a,b>0$ with $a+b=1$, find the minimum of $\dfrac{1}{a}+\dfrac{1}{b}$.

**Answer:** 4

**Method:** $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{a+b}{ab}=\dfrac{1}{ab}$. By AM–GM, $ab \leq \left(\dfrac{a+b}{2}\right)^2=\dfrac{1}{4}$, so $\dfrac{1}{ab}\geq 4$. Equality when $a=b=\frac{1}{2}$.

> **Investigate further:** Reprove the result via Cauchy–Schwarz: $\left(\dfrac{1}{a}+\dfrac{1}{b}\right)(a+b)\geq(1+1)^2=4$, and $a+b=1$. Generalise: for $a_1+a_2+\cdots+a_n=1$ with all $a_i>0$, find the minimum of $\displaystyle\sum_{i=1}^{n}\frac{1}{a_i}$ using the AM–HM inequality. (*Answer: $n^2$.*)

## Section D Challenge (TMUA / SMC difficulty)

**1.** Find all real solutions to $x^4 + 2x^3 - 2x - 1 = 0$.

**Answer:** $x = 1$ and $x = -1$ ($x = -1$ has multiplicity 3)

**Method:** Group into two differences: $x^4 + 2x^3 - 2x - 1 = (x^4 - 1) + 2x(x^2 - 1)$. Both terms share the factor $x^2 - 1$: $(x^2-1)(x^2+1)+2x(x^2-1) = (x^2-1)(x^2+2x+1) = (x-1)(x+1)(x+1)^2 = (x-1)(x+1)^3$. The only real solutions are $x = 1$ and $x = -1$.

> **Investigate further:** The factorisation $x^4 + 2x^3 - 2x - 1 = (x-1)(x+1)^3$ shows $x = -1$ is a root of multiplicity 3. Verify by computing $f(-1)$, $f'(-1)$, $f''(-1)$ and checking they are all zero. Geometrically, a root of multiplicity 3 means the curve $y = f(x)$ crosses the $x$-axis and has an inflection point there simultaneously. Explore: what does multiplicity 4 look like? Sketch $y = (x-a)^4$ near $x = a$.

**2.** Prove that $n^2 + n + 41$ is not always prime by finding the smallest counterexample, and explain why it is prime for $n = 0, 1, \ldots, 39$.

**Answer:** Smallest counterexample: $n = 40$, giving $40^2 + 40 + 41 = 41^2 = 1681 = 41 \times 41$.

**Method:** $n = 40$: $1600 + 40 + 41 = 1681 = 41^2$. Composite. □ For $0 \le n \le 39$: one approach is to observe that $f(n) = n^2 + n + 41$ — any factor $d$ of $f(n)$ with $d \le 40$ would divide $f(n) - f(0) = n(n+1)$ and also 41, forcing $d = 1$ or $d = 41$. Since $f(n) < 41^2$ for $n < 40$, the factor 41 cannot appear.

> **Investigate further: Euler's lucky numbers.** The polynomial $n^2 + n + p$ is prime for $n = 0, 1, \ldots, p-2$ if and only if $p$ is one of Euler's lucky numbers: $2, 3, 5, 11, 17, 41$. This connects to the theory of *imaginary quadratic fields* with class number 1. Verify that $n^2 + n + 17$ is prime for $n = 0, \ldots, 15$ and find the first counterexample. Then show directly that $n^2 + n + 41 \equiv 0 \pmod{41}$ iff $41 \mid n$ or $41 \mid (n+1)$.

**3.** Let $a, b, c > 0$ with $abc = 1$. Prove that $a + b + c \ge 3$.

**Answer:** Proof via AM–GM: see method.

**Method:** By AM–GM on three positive numbers: $\dfrac{a+b+c}{3} \ge \sqrt[3]{abc} = \sqrt[3]{1} = 1$. Therefore $a + b + c \ge 3$. Equality iff $a = b = c = 1$. □ This is the AM–GM inequality in its three-variable form, with the constraint doing all the work.

> **Investigate further: AM–GM for $n$ variables:** $\dfrac{a_1 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 \cdots a_n}$ for positive reals, with equality iff all are equal. Use it to prove: for $a, b, c > 0$ with $a + b + c = 3$, we have $abc \le 1$. (*This is the 'flipped' version of today's problem.*) Then prove: $a^2 + b^2 + c^2 \ge \dfrac{(a+b+c)^2}{3}$ (power-mean inequality) using $3(a^2 + b^2 + c^2) \ge (a+b+c)^2$ via the Cauchy–Schwarz inequality.

**4.** For positive reals $a, b$, prove $\dfrac{a}{b} + \dfrac{b}{a} \ge 2$. Hence prove AM–GM: $\dfrac{a+b}{2} \ge \sqrt{ab}$.

**Answer:** Proof: see method.

**Method:** *Part 1:* Let $t = \frac{a}{b} > 0$. Then $t + \frac{1}{t} - 2 = \dfrac{t^2 - 2t + 1}{t} = \dfrac{(t-1)^2}{t} \ge 0$. So $\frac{a}{b} + \frac{b}{a} \ge 2$, equality iff $a = b$. □

*Part 2:* Apply Part 1 to the positive reals $\sqrt{a}$ and $\sqrt{b}$: $\dfrac{\sqrt{a}}{\sqrt{b}} + \dfrac{\sqrt{b}}{\sqrt{a}} \ge 2 \Rightarrow \dfrac{a+b}{\sqrt{ab}} \ge 2 \Rightarrow \dfrac{a+b}{2} \ge \sqrt{ab}$.

□

> **Investigate further:** These two proofs show that *every algebraic inequality ultimately rests on the fact that squares are non-negative.* Find at least three distinct proofs of AM–GM for two variables. Then prove the *weighted AM–GM*: for $\lambda \in (0, 1)$, $\lambda a + (1 - \lambda)b \geq a^{\lambda} b^{1-\lambda}$. (*This generalises AM–GM and is fundamental in convex analysis.*)

**5.** Find the number of integers $n$ with $1 \leq n \leq 1000$ such that $(n + 3) \mid (n^2 - 1)$.

**Answer:** 2 ($n = 1$ and $n = 5$)

**Method:** $n^2 - 1 = (n+3)(n-3)+8$, so $(n+3) \mid (n^2 - 1)$ iff $(n+3) \mid 8$. Since $n \geq 1$, we need $n+3 \geq 4$, and the only divisors of 8 that are $\geq 4$ are 4 and 8, giving $n = 1$ and $n = 5$. Exactly **2** values in range.

> **Investigate further:** Generalise: find all $n \geq 1$ such that $(n + k) \mid (n^2 - 1)$ for fixed $k$. Write $n^2 - 1 = (n + k)(n - k) + (k^2 - 1)$, so the condition becomes $(n + k) \mid (k^2 - 1)$. This is a standard technique: *reduce the divisibility to a condition on a constant.* Use it to find all $n \geq 1$ with $(n + 2) \mid (n^2 + 1)$.

## Top 5 Patterns Today

1. **Remainder & Factor Theorem.** To find the remainder of $f(x) \div (x - a)$: evaluate $f(a)$. To show $(x - a)$ is a factor: show $f(a) = 0$. Instantaneous — no long division.
2. **AM–GM and the non-negativity of squares.** Every algebraic inequality ultimately reduces to $(x - y)^2 \geq 0$ or a sum-of-squares. AM–GM ($\frac{a+b}{2} \geq \sqrt{ab}$) is the compressed form of this for products.
3. **Completing the square in two (or more) variables.** $(x + h)^2 + (y + k)^2$ is the canonical form for optimisation. The minimum is the squared distance from a point to the origin after shifting.
4. **Rational Root Theorem.** For $a_n x^n + \cdots + a_0$, rational roots have the form $p/q$ with $p \mid a_0$, $q \mid a_n$. Combined with the Factor Theorem: test candidates, divide out, repeat.
5. **Divisibility via algebraic identity.** To show $(n + k) \mid f(n)$: write $f(n) = Q(n) \cdot (n + k) + R$ where $R$ is a constant. Then $(n + k) \mid f(n)$ iff $(n + k) \mid R$ — reducing an infinite problem to a finite one.

## Common Traps to Avoid

1. **Applying AM–GM to negative numbers.** $\frac{a+b}{2} \geq \sqrt{ab}$ requires $a, b > 0$. If $a < 0$, the square root $\sqrt{ab}$ may not even be real. Always check the sign condition before applying any mean inequality.
2. **Introducing extraneous solutions when squaring.** Squaring $\sqrt{A} = B$ requires $B \geq 0$ *first*. Always substitute back into the original equation to check.
3. **Forgetting that the Rational Root Theorem gives candidates, not roots.** Every candidate must be tested. Not every $p/q$ is a root — the theorem only rules out impossible rational roots, it does not guarantee any exist.
4. **Confusing divisibility direction.** $(n + k) \mid (n^2 - 1)$ does *not* mean $(n^2 - 1) \mid (n + k)$. Write out the algebra explicitly: $a \mid b$ means $b = ka$ for some integer $k$.

5. **Treating multiplicity-$k$ roots as $k$ distinct solutions.** $x = -1$ is a root of $(x+1)^3(x-1) = 0$ with multiplicity 3 but it is *one* value of $x$. Count solutions, not multiplicities, unless the question asks.

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*"Mathematics is not about numbers, equations or algorithms: it is about understanding."*
*— William Paul Thurston*

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