Daily Algebra Drill #1 — Answers & Investigations

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Section	**Topic**	**Questions**	**Target time**
A	Rapid Recognition	10	2 min 30 sec
B	Manipulation Drills	10	8 min
C	Substitution & Structure	8	10 min
D	Challenge	5	15 min

New toolkit today: Difference / Sum of Squares & Cubes · Newton's Identity Chain · Disguised Quadratics · Telescoping & Partial Fractions.

Section A Rapid Recognition ($\leq$15 s each)

Factorise, simplify, or evaluate instantly. No expansion. Pure recognition.

1. Factorise completely: $x^4 - 16$.

Answer: $(x^2+4)(x+2)(x-2)$

Method: Apply the difference of two squares twice: $(x^2)^2 - 4^2 = (x^2+4)(x^2-4) = (x^2+4)(x+2)(x-2)$.

Investigate further: Can you factorise $x^4 + 16$ over the real numbers? Hint: Add and subtract $8x^2$ to complete the square, creating a hidden difference of squares.

2. Simplify: $\dfrac{a^2 - b^2}{a^2 + 2ab + b^2}$.

Answer: $\dfrac{a-b}{a+b}$

Method: Factor both numerator and denominator: $\dfrac{(a-b)(a+b)}{(a+b)^2}$. Cancel out one factor of $(a+b)$.

Investigate further: What if the numerator was $a^3 - b^3$ and the denominator a^2+ab+b^2? Recognise how standard identities pair up to simplify fractions.

3. Without expanding, evaluate: $103^2 - 97^2$.

Answer: 1200

Method: Use difference of two squares: $(103+97)(103-97) = (200)(6) = 1200$. Always use this for close integers.

Investigate further: Use this concept backward: how can you mentally compute 103×97? Treat it as $(100+3)(100-3) = 10000 - 9$.

4. Factorise: $2x^2 + 7x - 15$.

Answer: $(2x-3)(x+5)$

Method: Look for factors of $2 \times -15 = -30$ that sum to 7. These are 10 and -3. Splitting the middle term and grouping yields the factors.

> **Investigate further:** Use the quadratic formula to find the roots $3/2$ and -5. Notice how the roots immediately dictate the $(2x-3)(x+5)$ structure via the Factor Theorem.

5. Simplify: $\left(x+\dfrac{1}{x}\right)^2 - \left(x-\dfrac{1}{x}\right)^2$.

Answer: 4

Method: Use the identity $(A+B)^2-(A-B)^2=4AB$. Here $A=x$ and $B=\frac{1}{x}$, so the result is $4\cdot x\cdot\frac{1}{x}=4$.

> **Investigate further:** This algebraic identity has a geometric proof involving the area of rectangles. Try drawing a square of side $(A+B)$ and removing a square of side $(A-B)$.

6. Factorise: x^3+8.

Answer: $(x+2)(x^2-2x+4)$

Method: Apply the sum of cubes identity: $a^3+b^3=(a+b)(a^2-ab+b^2)$ with $a=x$ and $b=2$.

> **Investigate further:** Why does the quadratic factor x^2-2x+4 never have real roots? Check its discriminant and understand how this applies to all sum/difference of cubes.

7. Simplify: $\dfrac{x^2-5x+6}{x^2-4}$.

Answer: $\dfrac{x-3}{x+2}$

Method: Factor the numerator as $(x-2)(x-3)$ and the denominator as $(x-2)(x+2)$. Cancel the common $(x-2)$ term.

> **Investigate further:** Sketch the graph of this function. The cancelled $(x-2)$ factor doesn't just disappear—it creates a "hole" (removable discontinuity) at $x=2$. What is the y-coordinate of this hole?

8. If $a+b=5$ and $ab=3$, find a^2+b^2.

Answer: 19

Method: Use the symmetric polynomial identity: $a^2+b^2=(a+b)^2-2ab=25-6=19$. Never compute a and b explicitly.

> **Investigate further:** Now find a^3+b^3 using the formula $(a+b)^3-3ab(a+b)$. You should be able to do this purely mentally.

9. Factorise: $4x^2-12x+9$.

Answer: $(2x-3)^2$

Method: Recognise the perfect square trinomial structure: $(ax-b)^2=a^2x^2-2abx+b^2$. Here $a=2$ and $b=3$.

> **Investigate further:** A quadratic is a perfect square if and only if its discriminant equals zero. Verify that $b^2-4ac=0$ for this expression.

10. Without a calculator, simplify: $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{9\cdot 10}$.

Answer: $\dfrac{9}{10}$

Method: Use partial fractions: $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$. The series telescopes to $(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})\cdots=1-\frac{1}{10}=\frac{9}{10}$.

> **Investigate further:** Generalise this to $\sum_{k=1}^{n}\frac{1}{k(k+2)}$. The partial fraction split is $\frac{1}{2}(\frac{1}{k}-\frac{1}{k+2})$, which creates a delayed telescoping effect.

Section B Manipulation Drills (≤50 s each)

Algebraic fractions, rearranging, hidden structure. Resist the urge to expand.

1. Simplify: $\dfrac{x^3-1}{x^2-1}$.

Answer: $\dfrac{x^2+x+1}{x+1}$

Method: Factor the numerator as a difference of cubes: $(x-1)(x^2+x+1)$. Factor the denominator as $(x-1)(x+1)$. Cancel $(x-1)$.

> **Investigate further:** This is $\frac{x^3-1}{x-1}\div\frac{x^2-1}{x-1}$. Recognise the geometric series polynomials: you are effectively dividing (x^2+x+1) by $(x+1)$.

2. Simplify: $\dfrac{1}{x-1}-\dfrac{2}{x^2-1}$.

Answer: $\dfrac{1}{x+1}$

Method: The common denominator is $(x-1)(x+1)$. Multiply the first fraction by $\frac{x+1}{x+1}$. The numerator becomes $(x+1)-2=x-1$, which perfectly cancels with the denominator.

> **Investigate further:** Try simplifying $\frac{1}{x-1}+\frac{1}{x+1}-\frac{2}{x^2-1}$. Grouping the terms creatively often reveals the cancellation faster.

3. Make x the subject: $\dfrac{x+a}{x-a}=k \quad (k\neq 1)$.

Answer: $x=a\dfrac{k+1}{k-1}$

Method: Cross-multiply: $x+a=k(x-a)$. Expand: $x+a=kx-ka$. Collect x terms: $x-kx=-ka-a$. Factor: $x(1-k)=-a(k+1) \implies x=a\frac{k+1}{k-1}$.

> **Investigate further:** This is a Möbius transformation. Notice that if $f(x)=\frac{x+a}{x-a}$, then the inverse function $f^{-1}(k)$ essentially has the exact same structural form!

4. Given that $a+b+c=0$, express $(a+b)(b+c)(c+a)$ in terms of abc.

Answer: $-abc$

Method: Use the constraint three times: $a+b=-c,\ b+c=-a,\ c+a=-b$. The product becomes $(-c)(-a)(-b)=-abc$. Constraint substitution beats expansion every time.

> **Investigate further:** The condition $a+b+c=0$ is famously linked to $a^3+b^3+c^3=3abc$. Try proving that identity by expanding $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

5. Show that $\dfrac{x^2+3x+2}{x^2-x-2}$ simplifies, and state all values of x for which the original expression is undefined.

Answer: $\dfrac{x+2}{x-2}$; undefined at $x=2$ and $x=-1$.

Method: Factor the numerator as $(x+1)(x+2)$ and denominator as $(x-2)(x+1)$. Cancel $(x+1)$. The original fraction is undefined wherever the original denominator is zero.

> **Investigate further:** What is the horizontal asymptote of this rational function? As $x\to\infty$, the ratio of the leading coefficients dictates the limit.

6. Simplify: $\left(\dfrac{x^2}{y}-\dfrac{y^2}{x}\right)\div\left(\dfrac{x}{y}+1+\dfrac{y}{x}\right)$.

Answer: $x-y$

Method: Write both brackets with common denominator xy.
First bracket: $\frac{x^3-y^3}{xy}$. Second bracket: $\frac{x^2+xy+y^2}{xy}$.
The xy denominators cancel, leaving $\frac{x^3-y^3}{x^2+xy+y^2}$, which is precisely the formula for $(x-y)$.

> **Investigate further:** What changes if the first bracket is $\frac{x^2}{y}+\frac{y^2}{x}$ instead? How does the second bracket need to adjust to result in $x+y$?

7. Factorise: $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

Answer: $-(a-b)(b-c)(c-a)$ or $(a-b)(c-b)(c-a)$

Method: By the Factor Theorem, if setting $a=b$ makes the expression zero, then $(a-b)$ is a factor. By cyclic symmetry, $(b-c)$ and $(c-a)$ are also factors. To check the sign, expand a test term: the given expression has $+a^2b$, but $(a-b)(b-c)(c-a)$ expands to $-a^2b$, hence the negative sign is required.

> **Investigate further:** This cyclic alternating polynomial is the determinant of a 3×3 Vandermonde matrix. Investigate what a Vandermonde matrix is and why its determinant factors so beautifully.

8. Solve for x: $\dfrac{3}{x+1}+\dfrac{4}{x+2}=\dfrac{11}{(x+1)(x+2)}$.

Answer: $x=\dfrac{1}{7}$

Method: Multiply the entire equation by the common denominator $(x+1)(x+2)$. This leaves a simple linear equation: $3(x+2)+4(x+1)=11 \implies 7x+10=11 \implies x=\frac{1}{7}$. Neither denominator vanishes at $x=\frac{1}{7}$, so the solution is valid.

> **Investigate further:** If the numerators were x instead of constants, you would end up with a quadratic equation. Always check if your final roots cause a division by zero in the original fractions!

9. Simplify: $\dfrac{2^{n+2}+2^n}{2^{n+1}+2^{n-1}}$.

Answer: 2

Method: Factor out the smallest power from both top and bottom.
Top: $2^n(2^2+1) = 2^n(5)$.
Bottom: $2^{n-1}(2^2+1) = 2^{n-1}(5)$.
The 5 cancels, leaving $2^n/2^{n-1} = 2^1 = 2$.

> **Investigate further:** Can you see that the numerator is exactly 2 times the denominator just by looking at the indices? $(n+2)$ is one higher than $(n+1)$, and n is one higher than $(n-1)$.

10. Given that $p+q=7$ and $p^3+q^3=133$, find pq.

Answer: $pq = 10$

Method: Use the identity $p^3+q^3 = (p+q)[(p+q)^2-3pq]$.
Substitute the knowns: $133 = 7[49-3pq]$. Divide by 7: $19 = 49-3pq$.
Rearrange: $3pq = 30 \implies pq = 10$.

> **Investigate further:** Now that you have $p+q=7$ and $pq=10$, you can construct the quadratic $x^2-7x+10=0$. Its roots are 2 and 5, revealing the underlying values of p and q effortlessly.

Section C Substitution & Structure (≤90 s each)

Look for symmetry, disguised quadratics, and useful substitutions before computing anything.

1. Solve: $x^4-5x^2+4=0$.

Answer: $x = \pm1,\ \pm2$

Method: This is a disguised quadratic. Let $u = x^2$. The equation becomes $u^2-5u+4=0$, which factorises to $(u-1)(u-4)=0$. Hence $x^2=1 \implies x=\pm1$ and $x^2=4 \implies x=\pm2$.

> **Investigate further:** What condition must k satisfy for $x^4-kx^2+4=0$ to have exactly four distinct real roots? (Answer: $k>4$). Analyse the discriminant and root positivity.

2. If $x+\dfrac{1}{x}=3$, find $x^3+\dfrac{1}{x^3}$.

Answer: 18

Method: Square the given equation: $x^2+2+\frac{1}{x^2}=9 \implies x^2+\frac{1}{x^2}=7$.
Now multiply $(x+\frac{1}{x})(x^2+\frac{1}{x^2}) = 3\times7 = 21$.
Expanding the left gives $x^3+\frac{1}{x}+x+\frac{1}{x^3}=21$. Substitute $x+\frac{1}{x}=3$ to get 18.

> **Investigate further:** Alternatively, memorise the direct identity t^3-3t where $t = x+\frac{1}{x}$. Use this to quickly evaluate $x^5+\frac{1}{x^5}$.

3. Find all real solutions to: $(x^2-3x)^2-2(x^2-3x)-8=0$.

Answer: $x = -1,\ 1,\ 2,\ 4$

Method: Substitute $u = x^2-3x$. The equation becomes $u^2-2u-8=0$, giving $(u-4)(u+2)=0$.
Case 1: $x^2-3x=4 \implies x^2-3x-4=0 \implies (x-4)(x+1)=0$.
Case 2: $x^2-3x=-2 \implies x^2-3x+2=0 \implies (x-2)(x-1)=0$.

> **Investigate further:** This highlights the power of chunking. Try solving $(x^2 - 3x + 1)^2 - 4(x^2 - 3x + 1) + 3 = 0$ using the exact same strategy.

4. Simplify: $\dfrac{x^2 + y^2}{xy} - \dfrac{(x - y)^2}{xy}$.

Answer: 2

Method: Since they share a denominator, combine them: $\frac{x^2+y^2-(x^2-2xy+y^2)}{xy}$. The squared terms perfectly cancel, leaving $\frac{2xy}{xy} = 2$.

> **Investigate further:** This implies $\frac{x^2+y^2}{xy} \geq 2$ because $(x - y)^2 \geq 0$ for real numbers. You have just proved a form of the AM-GM inequality!

5. The expression $x^4 + 4y^4$ can be factorised over the integers. Do so.

Answer: $(x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy)$

Method: Add and subtract $4x^2y^2$ to complete a square:
$x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 = (x^2 + 2y^2)^2 - (2xy)^2$.
Now apply difference of two squares to get the final answer.

> **Investigate further:** This is the famous Sophie Germain Identity. Use it to prove that $n^4 + 4$ is never a prime number for any integer $n > 1$.

6. Solve: $\sqrt{2x + 1} + \sqrt{2x - 1} = t$ for x in terms of t ($t > 0$).

Answer: $x = \dfrac{t^4 + 4}{8t^2}$

Method: Multiply the given equation by its conjugate: $(\sqrt{2x + 1} + \sqrt{2x - 1})(\sqrt{2x + 1} - \sqrt{2x - 1}) = (2x + 1) - (2x - 1) = 2$.
Thus, the conjugate $\sqrt{2x + 1} - \sqrt{2x - 1} = 2/t$.
Add the original equation and the conjugate: $2\sqrt{2x + 1} = t + 2/t = \frac{t^2+2}{t}$.
Square both sides: $4(2x + 1) = \frac{t^4+4t^2+4}{t^2} \implies 8x + 4 = t^2 + 4 + \frac{4}{t^2} \implies 8x = \frac{t^4+4}{t^2}$.

> **Investigate further:** The conjugate trick is extremely powerful for radical equations. Why is squaring the original equation directly $\cdots = t^2$ so much messier here?

7. Given that $a + b + c = 0$, evaluate $\dfrac{a^2}{bc} + \dfrac{b^2}{ca} + \dfrac{c^2}{ab}$.

Answer: 3

Method: Combine the fractions over the common denominator abc. The numerator becomes $a^3 + b^3 + c^3$. We know that when $a + b + c = 0$, $a^3 + b^3 + c^3 = 3abc$. So the expression is $\frac{3abc}{abc} = 3$.

> **Investigate further:** Investigate how to compute $a^4 + b^4 + c^4$ when $a + b + c = 0$. (Hint: square $a^2 + b^2 + c^2$ and use symmetric polynomials).

8. Solve the system: $x + y = 5, \quad x^2 + y^2 = 17$.

Answer: $(x, y) = (1, 4)$ or $(4, 1)$

Method: Use $(x+y)^2 = x^2+y^2+2xy \implies 25 = 17+2xy \implies xy = 4$.
Since $x+y=5$ and $xy=4$, x and y are the roots of the quadratic $t^2-5t+4=0$, which factors as $(t-1)(t-4)=0$.

> **Investigate further:** Geometrically, you just found the intersection points of the line $y=-x+5$ and the circle $x^2+y^2=17$. Sketch this to visually confirm the symmetric coordinates.

Section D Challenge (TMUA / SMC difficulty)

Insight over computation. Each question has a short, elegant solution — find it.

1. Find the value of

$$\frac{(2025)^3-(2024)^3-(2023)^3+(2022)^3}{(2025)^2-(2022)^2}.$$

Answer: 2

Method: Let $x=2022$. The expression is $\frac{(x+3)^3-(x+2)^3-(x+1)^3+x^3}{(x+3)^2-x^2}$.
Pair the terms with matching signs:
$(x+3)^3-(x+2)^3 = 3x^2+15x+19$.
$x^3-(x+1)^3 = -3x^2-3x-1$.
Adding: the numerator is $12x+18$.
Denominator: $(x+3)^2-x^2 = 6x+9$. The ratio is exactly $\frac{2(6x+9)}{6x+9}=2$.

> **Investigate further:** Notice how the x^3 and x^2 terms completely vanish. This means for ANY sequence of four consecutive integers a,b,c,d, the ratio $\frac{d^3-c^3-b^3+a^3}{d^2-a^2}$ is a constant. Prove it.

2. Let $f(x)=\dfrac{x}{x+1}$. Define f^n as f composed with itself n times. Find a closed form for $f^n(x)$ and hence compute $f^{2025}(1)$.

Answer: $f^n(x)=\dfrac{x}{nx+1}$; $f^{2025}(1)=\dfrac{1}{2026}$

Method: Compute the first few iterates to spot the pattern:
$f^2(x) = f\left(\frac{x}{x+1}\right) = \frac{\frac{x}{x+1}}{\frac{x}{x+1}+1} = \frac{x}{x+(x+1)} = \frac{x}{2x+1}$.
$f^3(x) = \frac{\frac{x}{2x+1}}{\frac{x}{2x+1}+1} = \frac{x}{3x+1}$.
By induction, $f^n(x)=\frac{x}{nx+1}$. Therefore $f^{2025}(1)=\frac{1}{2025(1)+1}=\frac{1}{2026}$.

> **Investigate further:** Rational functions of the form $\frac{ax+b}{cx+d}$ are Möbius transformations. They can be composed by multiplying 2×2 matrices! Try representing $f(x)$ as a matrix $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ and finding its n-th power.

3. Positive reals a,b,c satisfy $abc=1$. Simplify:

$$\frac{1}{1+a+ab}+\frac{1}{1+b+bc}+\frac{1}{1+c+ca}.$$

Answer: 1

Method: Multiply the numerator and denominator of the second fraction by a: $\frac{a}{a+ab+abc}=\frac{a}{a+ab+1}$.
Multiply the numerator and denominator of the third fraction by ab: $\frac{ab}{ab+abc+a^2bc}=\frac{ab}{ab+1+a}$.
All three fractions now share the exact same denominator $1+a+ab$.
The numerators sum to $1+a+ab$, making the entire expression equal to 1.

> **Investigate further:** Try generalising this trick for 4 variables a, b, c, d where $abcd = 1$. What do the denominators look like?

4. Without solving for x, find the value of $x^2 + \dfrac{1}{x^2}$ if $x - \dfrac{1}{x} = \sqrt{5}$. Hence find the exact value of $x^4 + \dfrac{1}{x^4}$.

Answer: $x^2 + \dfrac{1}{x^2} = 7$; $\quad x^4 + \dfrac{1}{x^4} = 47$

Method: Square both sides: $(x - \frac{1}{x})^2 = x^2 - 2 + \frac{1}{x^2} = (\sqrt{5})^2 = 5$.
Add 2: $x^2 + \frac{1}{x^2} = 7$.
Square this new relation: $(x^2 + \frac{1}{x^2})^2 = x^4 + 2 + \frac{1}{x^4} = 7^2 = 49$.
Subtract 2: $x^4 + \frac{1}{x^4} = 47$.

> **Investigate further:** Use this upward-building technique to evaluate $x^3 - \frac{1}{x^3}$ starting from the original given equation. Remember the minus sign!

5. Find all integers n such that $n^2 + 4n + 3$ is a perfect square.

Answer: $n = -1, \ -3$

Method: Complete the square: $n^2 + 4n + 3 = (n + 2)^2 - 1$.
We want this to equal some perfect integer square m^2.
$(n + 2)^2 - m^2 = 1$. This is a difference of two squares: $(n + 2 - m)(n + 2 + m) = 1$.
Since we are dealing with integers, the only integer factors of 1 are $(1)(1)$ and $(-1)(-1)$.
This forces $m = 0$, meaning $(n + 2)^2 = 1 \implies n + 2 = \pm 1$.
So $n = -1$ or $n = -3$. (Both yield 0, which is 0^2).

> **Investigate further:** Equations of the form $x^2 - dy^2 = 1$ are known as Pell's Equations. Here $d = 1$ makes it trivial, but what if you needed $n^2 + 4n + 2 = m^2$? Investigating this opens up a beautiful branch of Number Theory.

Top 5 Patterns Today

1. **Difference / Sum of Squares & Cubes.** $(a^2 - b^2)$, $(a^3 \pm b^3)$ — always factor these before executing any other algebraic operations. It reveals the structure immediately.

2. **Newton's Identity Chain.** If you know $p + q$ and pq, you can find $p^n + q^n$ by building upward iteratively. You never need to find the explicit (often messy radical) values of p or q.

3. **Disguised Quadratics.** When an expression has the form $[f(x)]^2 + b \cdot f(x) + c = 0$, substitute $u = f(x)$ immediately. Do not expand it into a higher-degree polynomial.

4. **Telescoping & Partial Fractions.** Products or sums of consecutive integers almost always telescope. Look for splits like $\frac{1}{n} - \frac{1}{n+1}$ to collapse the middle terms into oblivion.

5. **Constraint Exploitation.** When a constraint is given (e.g., $a + b + c = 0$ or $abc = 1$), it is *always* there to be used. Use it to substitute and collapse fractions before attempting brute-force common denominators.

Common Traps to Avoid

1. **Expanding when you should factor.** Expanding $x^4 - 16$ or multiplying out $(x+1)(x+3)$ before recognising a structure wastes time and completely obscures the elegant path to the answer.

2. **Solving for individual variables.** In systems like $p+q=7, p^3+q^3=133$, trying to solve for p explicitly via substitution is a trap. Always use symmetric polynomial properties to manipulate the block forms.

3. **Ignoring domain restrictions after cancellation.** In B5, the cancelled factor $(x+1)$ creates a hole at $x=-1$. Even though it vanishes from the algebra, it remains a restriction on the domain of the original expression.

4. **Assuming $a^3+b^3+c^3=(a+b+c)^3$ when $a+b+c=0$.** The correct identity is $a^3+b^3+c^3=3abc$ when the sum is zero. The cube of the sum is always zero, which is absolutely not the same as the sum of the cubes!

5. **Missing the elegant substitution in D-level questions.** Before writing any working down, ask: *What is the structure? Can I substitute? Is there a constraint I haven't used?* Every D-level question is designed to have a route under 4 lines.

"Speed comes from recognition, not from rushing. Every shortcut is a pattern you have internalised."

Found a mistake or want to contribute a sheet?
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